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Search Models Of Money

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by

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ABSTRACT

This thesis contains three essays studying the emergence of money as a medium of exchange. The search framework used assumes it is difficult for agents to come together in order to trade. The result is a double coincidence of wants problem that can be alleviated through the use of money as a medium of exchange. Much of the recent work studying search models of money is based on papers by Kiyotaki and Wright. This thesis extends their work to make this area of monetary economics more applicable.

Chapter 1 of the thesis introduces a credit instrument into the search model to show that, unlike many other monetary models, money and credit can coexist in equilibrium. Under certain conditions, agents will trade using both money and credit if they believe that other agents will also. Both money and credit may be welfare improving when the double coincidence of wants problem is sufficiently bad.

Chapter 2 relaxes some of the more restrictive inventory assumptions in the basic model and introduces an endogenous price determination mechanism. By allowing agents to store multiple units of money it is possible to introduce a Nash bargaining game between agents to determine the price at which they will trade in any match. When agents are restricted to hold finite levels of indivisible units, money will not be neutral. Only if agents are completely unrestricted in their holdings of money will money be neutral.

In the third chapter agents are permitted to produce the medium of exchange themselves. This model attempts to formalize within a search model some of the characteristics, described initially by Jevons, that a money should exhibit. It is found that

only commodities that are more costly to produce than consumption goods can circulate as generally accepted media of exchange. Similarly, only commodities that are sufficiently durable and portable will function as money. Given that agents tend to overproduce money, there is a role for government to either tax the production of the commodity money or to introduce a fiat money to reduce the amount of the private money in circulation.

DEDICATION

To my Poppa, who did not get to see me finish.

And to Carolyn, who has helped keep me sane throughout this insane process.

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INTRODUCTION:

This thesis is comprised of three essays that attempt to expand the area of monetary economics that studies the emergence of money as a medium of exchange. The models employ a search theoretic framework to generate trading frictions which can be alleviated through the use of money as a medium of exchange. In this type of search model, each type of agent produces a certain good but wishes to consume other brands of goods. The agents must then search for trading partners with whom they can form a double coincidence of wants and trade using barter. If money is introduced into this environment then it may emerge as a generally accepted medium of exchange because it can provide a faster and easier method of trading than barter.

The recent path breaking work in this area was provided by Nobuhiro Kiyotaki and Randall Wright (1989, 1990, 1991). Their work has built upon an earlier paper by Robert Jones (1976) and ultimately upon ideas put forth by Jevons (1877) and Menger in the last century. The Kiyotaki and Wright papers have shown that search models can be used to model the emergence of money as a medium of exchange. This thesis will extend their work in a number of directions so as to make this area of monetary economics more widely applicable.

The appeal of the search model of money is that, unlike many other common monetary models, it can explain at a fundamental level why money may emerge as a generally accepted medium of exchange. The model thus has much potential for examining many monetary policy questions that have been previously considered only within an environment with an exogenous money. There is much work that must still be completed however before some of these questions can be asked. This thesis is an attempt to expand the applicability of the search model of money so that it is ready to answer some of these questions.

Consider for a moment why the search model may be an improvement upon other types of monetary models that are commonly in use. The cash-in-advance (CIA) model is commonly applied in monetary economics and has proven to be quite useful for examining policy questions. That model, however, makes no attempt to explain why agents choose to trade using money and in fact assumes that individuals can only trade through the use of money.

An overlapping generations model (OLG) is also quite widely used in monetary economics. In this environment, money serves as a store of value and is utilized because of the temporal separation of agents. Money is the only means by which agents of different generations may trade. Credit is not possible between members of different generations in the two-period lived OLG model. Credit is possible between the heterogeneous agents within each generation. This credit will, however, either be a perfect substitute for money or it will drive money out of the economy. Both of these predictions are of course contrary to actual observation.

The Townsend turnpike model introduces a spatial separation of agents to close down private loan markets and yield a role for money as a medium of exchange. This model generates an endogenous money but has generally ignored the question of credit.

None of these other models is particularly satisfactory for considering the interaction between money and credit in the economy. Chapter 1 of this thesis emerged from this observation and proposes the use of the search model to analyze money and credit. These other monetary models also do not consider the characteristics which money must possess if it is to circulate as a medium of exchange. This issue is considered in Chapter 3. The second chapter of the thesis introduces a price determination mechanism into the search model so that agents endogenously set the prices at which they will trade.

Chapter 1 of the thesis, entitled "Credit in a Search Model with Money as a Medium of Exchange," introduces credit into the search model to show that, unlike many other monetary models, money and credit can coexist in an equilibrium of this model. The cash-in-advance models do not explicitly attempt to model any of the factors that will lead to money emerging as an endogenous medium of exchange. These models impose a value for money by assuming it must be used on one side of every trade and, generally, do not involve any use of credit. Overlapping generations models motivate the use of money through the temporal separation of generations of agents. However, in these models, credit will either drive money out of the economy or be a perfect substitute for it in equilibrium. The search model in this thesis provides a framework that motivates money as an endogenous generally accepted medium of exchange and introduces credit to serve as an alternative trading method which is not a perfect substitute for money.

Chapter 2, "Prices in a Search Model of Money," relaxes some of the more restrictive inventory assumptions in the basic model in order to introduce an endogenous price determination mechanism. By allowing agents to store multiple units of the medium of exchange it is possible to introduce a Nash bargaining game between agents to determine the price at which they will trade in any match. When agents are restricted to hold finite levels of an indivisible money instrument then money will not be neutral. Only if agents are completely unrestricted in their holdings of money will money be neutral.

The third chapter, "Endogenous Money and Goods Production in a Search Model," introduces the endogenous production of commodity money and goods by agents so that the model can be used to examine the relationship between money supply and output as well as the characteristics of commodities that serve as media of exchange. This model attempts to formalize with a rigorous model some of the characteristics, described initially by Jevons, which a commodity must exhibit if it is to be used as money. Kiyotaki and Wright (1989) examined the storage cost aspect of money while this model extends the analysis to consider also the importance of the production costs and durability of money. It is found that only commodities which are more costly to produce than consumption goods can potentially circulate as media of exchange. In an economy with multiple media of exchange, no commodity money will emerge which has strictly greater costs of use than the other circulating monies. A commodity money can coexist with a government supplied fiat money that has strictly smaller costs since agents are able to immediately produce the commodity money themselves and thereby guarantee it is available for use in trading while avoiding spending the time required to find an agent with fiat currency.

CHAPTER 1: CREDIT IN A SEARCH MODEL WITH MONEY AS A MEDIUM OF EXCHANGE

1.1 Introduction:

Modern economies make extensive use of both money and credit in order to help agents trade with one another. There must be models, therefore, which can explain the simultaneous use of both these instruments. The objective of this chapter is thus to design a model in which the use of money and credit in trades arises endogenously and simultaneously in equilibrium. A search theoretic framework is used to generate trading frictions which can be alleviated by the use of either money or credit. The addition of credit to a search model will add further support to this method of modelling money which has recently emerged in the literature.

Models of money often begin with some form of heterogeneity of agents with respect to preferences or endowments as a method of generating a desire for exchange. As a result of this heterogeneous population, agents will attempt to trade using credit. If, either by assumption or through some form of friction in the environment, these loan markets are prevented from opening, then fiat money may have value and be used to facilitate trade.

The cash-in-advance (CIA) model (Sargent, 1987) does not explicitly model any of these heterogeneities or frictions to motivate money but instead gives money its value by simply assuming money must be used on one side of all transactions. With a binding CIA constraint, it is implicitly assumed that certain forms of intermediation are prohibited so that agents cannot arbitrage away any interest differentials. The basic CIA model, thus, does not generally include credit. Lucas and Stokey (1987) apply the CIA constraint to only a certain fraction of transactions and thus create both 'cash' and 'credit' goods. However, the assumptions on the monetary side of the model imply that it is not entirely satisfactory as a model of money and credit.

An overlapping generations (OLG) model (see Sargent, 1987 or Blanchard and Fischer, 1989) gives money value by creating frictions through the temporal separation of agents. This closes down inter-generational loan markets and allows money to be used for trade between generations. There may be 'local' private loan markets in the OLG model if there are heterogenous agents within each generation. However, in the basic form of this model, credit will either drive money out of the economy (if the interest rate is too high) or be a perfect substitute for money. Both of these predictions are contrary to observations. However, these models can be modified so that there will be a rate of return dominance of credit over money. For instance, if money is placed in the utility function then it is possible that both money and credit will be used despite credit yielding a strictly greater return.

A Townsend turnpike model, however, uses the spatial separation of agents to close down private loan markets and yield a role for money as a medium of exchange.

This model is able to provide an explanation for the use of money but previous work has generally ignored the question of credit. Recently, Manuelli and Sargent (1988) have experimented with 'local' private loan markets in models in which agents move along the turnpike only once every few periods.

More recent work, however, has used search models to motivate endogenously the use of money as a medium of exchange. Fiat and commodity monies are able to alleviate trading frictions caused by the double coincidence of wants problems surrounding barter trade. Beliefs regarding the acceptability of money are crucial in these models in deciding whether money will or will not be valued in equilibrium. Kiyotaki and Wright (1989, 1990, 1991) have successfully modelled money using this search theoretic framework and have shown how a good may emerge as a commodity money and how fiat money may be valued due to its low storage cost and a general belief in its universal acceptability.

As stated, previous models generally removed credit either by assumption or through a specialized environment. Similarly, a search model (Kiyotaki and Wright, 1989, 1990, 1991) in which agents must look for trading partners usually rules out the possibility of credit. With a continuum of agents, the search framework implies that there is a zero probability of ever meeting the same agent twice. Hence, the authors argue there can be no credit because borrowers would be able to avoid their creditors forever and need never repay the loan. Knowing this, agents would never lend and credit markets would not open. The model in this chapter, however, shall assume the environment is such that the credit market may remain open despite the search frictions.

In contrast to the Kiyotaki and Wright papers, Diamond (1990) has agents who must search for trading partners but for whom credit is assumed to be still available. He assumes that borrowers can only avoid repaying their loans by forever dropping out of the economy. This constitutes such a severe penalty that debtors find repayment to be optimal. Diamond, however, does not consider the introduction of money to his economy concentrating instead on the issues of credit availability.

In reality, however, the use of both money and credit is prevalent in modern society. The model herein, therefore, attempts to explain the existence of both money and credit in equilibrium. The search model used combines the basic ideas of Kiyotaki and Wright (1990) and Diamond (1990). The principle role for money in this economy is as a medium of exchange. Certain credit markets are assumed to be open, despite the search frictions, due to environmental characteristics similar to those in Diamond (1990). Four pure strategy equilibria exist in which agents trade using different combinations of the available instruments: i) barter, ii) barter and money, iii) barter and credit, or iv) barter, credit, and money. These are Nash equilibria with rational expectations so the beliefs of the agents will help determine the outcome and vice versa. Certain parameter restrictions are also necessary to ensure the existence of particular equilibria. Specifically, the relative arrival rate of trading partners and production opportunities is crucial in deciding which type of equilibrium will occur. If agents believe that money and credit will be used in equilibrium and if the arrival rate of production opportunities is sufficiently high then there will be an equilibrium with endogenous money and credit.

There will also be a number of mixed strategy equilibria in the model. One such equilibrium can be considered to be an example of credit rationing. In contrast to recent literature emphasizing informational imperfections, this credit rationing equilibrium is the result of agents simply believing that that will be the outcome.

Since I first wrote this chapter, Shouyong Shi (1993) has also begun to work on introducing credit into the search environment. He has expanded on the work contained within my thesis by introducing a divisible consumption commodity. This permits him to examine the purchasing power of credit and money and to show once again that the search model can generate a rate of return dominance of credit over money. His model predicts the general existence of two equilibria with both money and credit being used for trading. One equilibrium yields greater purchasing power for the credit instrument in relation to money while the other predicts that money has higher purchasing power.

The model is described in Section 1.2 while Sections 1.3 through 1.6 relate the details of the various pure strategy equilibria. Section 1.7 provides some numerical simulations of the pure strategy equilibria while Section 1.8 describes the mixed strategy credit rationing equilibrium. The final section provides a summary and conclusion for the chapter.

1.2 The Model:

The model used in this chapter resembles that of Kiyotaki and Wright (1990). Certain modifications have been made to allow for the introduction of a credit mechanism into their model. As mentioned, Peter Diamond (1990) designed a search model with credit but did not introduce money into the environment. This model is an attempt to join some of the basic ideas of the two models.

There is a continuum of differentiated products indexed by the points on a circle of circumference two. There is also a unit mass of agents, where an agent is indexed by a point on the same circle. The agents and products are uniformly distributed on the circle.

Agents at point i on the circle most prefer good i and derive utility u(z) from a good of distance z from i. For simplicity, assume

$$u(z) = \begin{cases} u>0 & \text{for } z \leq x \\ 0 & \text{for } z>x \end{cases}$$

so that agents receive a fixed level of utility u from any good within an exogenously specified distance $0<x\leq 1$ from their preferred good. Goods within x of an agent's type can be considered consumable while commodities further away are non-consumable. Since the distance between i and any randomly selected good is uniformly distributed, the probability that agent i will want to consume that good is x. The value functions for agents will be independent of the particular types of the goods since u is independent of the distance z.

There are no direct utility costs of storing goods, fiat money, or credit instruments. However, an agent can only store one unit of one object at any given time. Therefore, an agent holding an IOU for future production cannot also hold money or goods. The inventory restriction implies an opportunity cost of holding one item versus another. All items are indivisible and there is free disposal of any commodity.

Agents with nothing in storage may produce commodities according to a Poisson process with a constant arrival rate of $\alpha>0$. Each production opportunity that is undertaken yields one unit of the agent's production good. A production good is indexed by a point on the same circle as are agent types. The production set is also uniformly distributed on the circle and an agent's production good and consumption type are independently distributed. An agent's type, ie. production and consumption good, is assumed to be public information. This ensures that all trades will involve a double coincidence of wants between trading partners. Production costs are normalized to zero. It is also assumed that agents are not able to consume their own production good and thus must search for trading partners.

Potential trading partners are met pairwise according to a Poisson process with a constant arrival rate of $\beta>0$. The major difference from the Kiyotaki and Wright model is that production and trading markets are not separated and, therefore, agents in this economy are able to search for trading partners at the same time as they await a production opportunity. This allows for meeting, and the possibility of a credit arrangement, between an agent who has already produced and one who has yet to produce.

There are transaction costs that must be incurred to be able to trade in this economy. When a real commodity is accepted in trade, a cost of ε (where $0<\varepsilon<u$), in utility terms, is paid by the receiver. This ensures that an agent will only trade his good for another if the new good yields utility. Hence, there will be no commodity money in this model. There is no transaction cost if money is received. There would still be an equilibrium if the model was expanded to allow for such a cost, even in excess of ε, but

this would merely be an unnecessary complication. Accepting an IOU for a future good involves bearing the same cost ε in utility terms and is paid when the IOU comes due ie. repayment is received. Equilibria would also exist with credit being used even if it involved a transaction cost for IOUs in excess of ε.

When two agents meet, both of whom have goods, there will be an exchange of these goods only if there is a double coincidence of wants. Since agents only receive positive utility from goods within x of their preferred good, the probability that a randomly selected trading partner has a good which is desirable is also x. The probability of a double coincidence of wants is thus x^2. If trade does occur, then both agents receive $u-\varepsilon$ of utility net of transaction costs. If trade is refused by one or both agents, then the trading match dissolves and both agents return to the trading process to search for new partners. It is not possible to maintain any connection with previous trading partners (except if an IOU has been exchanged).

If one agent in a trading match has money while the other possesses a good, then trade may occur if the money holder desires the other's good (probability x) and if the goods trader desires money (probability π). In accepting the good, the money holder must pay a cost of ε to receive the utility u.

When a goods trader meets someone with goods-in-process, credit may be used to complete a trade. If so, the existing good is immediately transferred from the lender and is consumed by the borrower for a net utility of $u-\varepsilon$. While the lender holds the borrower's IOU, he is assumed to be restricted from producing, as when a good or money is held. The borrower continues to search for a production opportunity and, when it is found, the resulting good is repaid to the lender and the credit contract is fulfilled. No further contact between the borrower and lender is possible. Since a borrower's production good is public information, this type of credit trade will only occur if there is a double coincidence of wants. The probability of such an event is again x^2. These credit arrangements will be beneficial to the lender if the expected payoff to trading is greater than the payoff to waiting for another trading partner. The borrower, of course, benefits by receiving immediate consumption.

Agents are assumed not to be able to borrow money since this would imply the borrower would be holding money while searching for production opportunities and thus may end up with both money and a good.[1]

After the initial meeting between the lender and borrower, the only subsequent contact is assumed to be restricted to the moment of production. This allows for the repayment of the loan while preventing the emergence of any long term trading arrangements between the two agents. Assume that, at the moment of production, the new good is immediately repaid to the lender (ie. that repayment is incentive compatible). Diamond assumes a more involved punishment mechanism to ensure that there is repayment. Simply assuming repayment achieves the same result with little loss in generality.

Finally, assume that IOU's cannot be traded and that a debtor can only repay his loans with a good that he produces. This latter assumption is merely a restatement of the

[1] While the borrower would in fact never simultaneously hold both money and a good, the good being instantly repaid to the lender upon production, this assumption is useful in simplifying the analysis.

assumption that the only subsequent contact between a borrower and lender is at the moment of production. Also assume that agents who are borrowers cannot also be lenders. This could likely be justified by assuming monitoring costs are prohibitively high when an agent is both a borrower and a lender.

An agent will fall into one of five groups: i) a goods trader (g), ii) a money trader (m), iii) a non-debtor producer (n), iv) a debtor producer (d), or v) a lender (l). A certain fraction of the population is initially endowed with a fixed supply of unbacked government fiat money, $0 \leq M \leq 1$. In any steady state equilibrium with valued money, the fraction of money traders will equal the exogenous money supply, ie. m=M.

Equations (1.1) to (1.5) below outline the flow value functions (or continuous time discounted Bellman equations) to the agents of being in one of the five states. The value of being in a particular state is dependent upon the opportunities that exist for the agent while in that state. From such opportunities the agent receives a flow dividend and a capital gain (or loss) in utility terms. Let V_i represent the steady state value an agent receives while in state i. Let $r>0$ represent the subjective rate of time preference.

$$\begin{aligned} rV_g &= \beta g x^2(u-\epsilon + V_n - V_g) + \beta m x \max_{\pi} \pi(V_m - V_g) \\ &\quad + \beta n x^2 \Theta \max_{\psi} \psi(V_l - V_g) \end{aligned} \tag{1.1}$$

Let V_g (equation (1.1)) represent the value to a goods trading agent of holding a good while looking for a trading opportunity. Trading partners for this goods trader arrive at the rate β. With probability gx^2 the agent who arrives is holding a good (g) and there is a double coincidence of wants (x^2). The goods are exchanged and the agent receives a net flow utility of u-ε before becoming a non-debtor producer (V_n). The second term on the right hand side of equation (1.1) represents the results of a meeting between the goods trader and a money trader who desires his good. Such a meeting occurs with probability mx. The goods trader optimizes by choosing the trading strategy π, the probability with which he becomes a money trader (V_m). Finally, there is a probability $nx^2\Theta$ that the arriving trading partner is a non-debtor producer (n), with whom there is a double coincidence of wants (x^2), and who wishes to borrow (probability Θ). The goods trader then chooses a trading strategy ψ, representing his probability of lending, in order to maximize the return.

$$rV_m = \beta g\Pi x(u-\epsilon + V_n - V_m) \tag{1.2}$$

Equation (1.2) represents the value to an agent of holding money (V_m). This type of agent will also have trading partners arrive at rate β. With probability $g\Pi x$, the arriving partner is a goods trader who wants money and has a good the money trader desires. Trade then occurs and the money trader receives the flow dividend u-ε and moves into the non-debtor producer state (V_n).

$$rV_n = \alpha(V_g - V_n) + \beta g x^2 \Psi \max_{\theta} \theta(u-\epsilon + V_d - V_n) \tag{1.3}$$

The value function for non-debtor agents is given by V_n in equation (1.3). These agents are in the midst of production and have opportunities arriving at rate α. After producing, they become goods traders and earn a capital gain V_g-V_n>0. When trading partners arrive at rate β, there is a probability $gx^2\Psi$ that the partner is a goods trader (g) who desires the good yet to be produced ($x\Psi$) and who has a good consumable by the producer (x). The producer then chooses the trading strategy θ, the probability of borrowing. Borrowers receive a good which is immediately consumed yielding a flow utility of u-ε. They then move into the debtor state, V_d, and work to repay the loan.

$$rV_d = \alpha(V_n - V_d) \tag{1.4}$$

Those producers in debt have a value function expressed by V_d in equation (1.4). The only opportunity open to the agent is to produce a good and repay it to the lender. It is assumed that the environment is such that repayment is either the optimal, or only, course of action available to the agent. When a production opportunity arrives, the good is conveyed to the lender and the debtor returns to the production process as a non-debtor producer V_n.

$$rV_l = \alpha(u-\epsilon+V_n - V_l) \tag{1.5}$$

Equation (1.5) gives V_l, the value function for goods lenders. Production opportunities for the borrower and, therefore repayment for the lender, arrive at the rate α. The lender receives the payoff u-ε net of transaction costs and becomes a non-debtor producer. Since production type is public information, loans will only be made to those producers with whom there is a double coincidence of wants. Hence, the lender will alw_ys desire the good that is ultimately produced so there is no uncertainty as to the lender's return. If production type were private information the lender would face a possibility of a loss upon repayment if a non-consumable good were produced.

The population fractions will be governed by the differential equations given below. Steady state levels of these fractions are found by setting the rates of change to zero and using the equilibrium condition m=M and the identity 1=g+m+n+d+l.

$$\dot{g} = \alpha n - \beta g^2 x^2 - \beta gmx\pi - \beta gnx^2\theta\psi \tag{1.6}$$

$$\dot{m} = \beta gmx\pi - \beta mgx\pi = 0 \tag{1.7}$$

$$\dot{n} = \beta g^2 x^2 + \beta mgx\pi + \alpha d + \alpha l - \alpha n - \beta ngx^2\psi\theta \tag{1.8}$$

$$\dot{d} = \beta ngx^2\psi\theta - \alpha d \tag{1.9}$$

$$\dot{l} = \beta gnx^2\theta\psi - \alpha l \tag{1.10}$$

Equation (1.6) is the law of motion governing the fraction of the population with goods. The number of agents with a good to trade (g) will increase when a production opportunity arrives for a non-debtor producer (αn). The fraction g will decline if a goods trader meets another goods trader and there is a double coincidence of wants. If a goods trader meets an agent with money and both wish to trade, then g will fall again. The final term in equation (1.6) implies that g will also decline if a goods trader meets a non-debtor producer (probability gn) when there is a double coincidence of wants (probability x^2) and a desire to borrow and lend (probability $\theta\psi$). Equation (1.7) implies that there is no tendency for m to change, even when not in steady state.

The number of non-debtor producers (n) increases after a double coincidence of wants goods trade, when a money trader meets a goods trader, and when a loan is repaid. The fraction n decreases when a non-debtor produces or when they borrow. The number of debtors (or lenders) increases when a non-debtor producer meets a goods trader and there is a double coincidence of wants. This fraction falls when a debtor repays. Obviously, the number of debtors will equal the number of lenders in this model.

Using the identities m=M and 1=g+m+n+d+l and imposing steady state (ie. the rate of change of the population fractions is set to zero) will yield a system of equations which can be solved for the steady state levels of the population fractions.

The sections that follow discuss the pure strategy equilibria that exist for the model described above. Beginning with the simplest equilibrium, Section II discusses the pure barter equilibrium. Sections III and IV describe the barter/money and barter/credit equilibria, respectively, while Section V outlines a full equilibrium with barter, money, and credit being used.

1.3 Barter Equilibrium:

There will be a barter equilibrium in which neither money nor credit is used in conducting trades. All money that was initially held is disposed of immediately. Trade will only occur if both agents have already produced and there is a double coincidence of wants. No agent would give up a current good or his future production opportunity in exchange for money while believing that no other agent would accept that money in a subsequent trade. If it were also true that agents believed that credit would never be used and that the return to lending was negative, then there would be a pure barter equilibrium.

Suppose agents believe that other agents set $\Pi=0$, $\Theta=1$, and $\Psi=0$. After imposing these beliefs on equations (1.1) to (1.5), some straightforward algebra reveals the following expressions for the payoffs to money trading and the use of credit.

$$r(V_m - V_g) = -\beta g x^2 (u - \epsilon + V_n - V_g) < 0 \tag{1.11}$$

$$(r+\alpha)(V_l - V_g) = (\alpha - \beta g x^2)(u - \epsilon + V_n - V_g) \tag{1.12}$$

$$(r+\alpha)(u - \epsilon + V_d - V_n) = r(u - \epsilon) + \alpha(u - \epsilon + V_n - V_g) > 0 \tag{1.13}$$

From equation (1.11) we can see that no one will take money in trade since beliefs have ensured that there is a loss to accepting money[2], ie. V_m-V_g<0. Agents who initially held money would immediately dispose of it and begin producing. Equation (1.13) reveals that non-debtor producers are still willing to borrow goods, despite their belief that no one will lend, so a barter equilibrium requires a parameter restriction ensuring that no agent will lend. The return to lending, V_l-V_g, given in equation (1.12) may be either positive or negative. If the parameters α, β and x are such that $\alpha<\beta g x^2$ then the payoff to lending will be negative. This restriction basically states that, in order for lending to be unprofitable, the arrival rate of a consumable good through lending (α) must be less than the arrival rate of goods trading partners with a double coincidence of wants ($\beta g x^2$).

Therefore, if agents believe that others are following strategies of $\Pi=0$, $\Theta=1$, and

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[2] See the appendix for a proof that u-ε+V_n-V_g>0.

Ψ=0 then, given $\alpha<\beta gx^2$, they will find it optimal to set the same strategies. A barter equilibrium will then result. Note that the belief that money will not be used in trade is enough to ensure no one will trade for money. However, an agent's belief that no one else will lend is not sufficient to ensure that the agent himself will not lend if given the opportunity. Only with an additional parameter restriction will the lending return be negative and the agents' strategy choice be consistent with their beliefs.

1.4 Barter/Money Equilibrium:

The model may also have an equilibrium in which both money and barter are used trading. If all agents believe that no one will participate in credit trades and certain parameter restrictions hold then an equilibrium of this type may occur. This non-credit equilibrium is essentially the Kiyotaki and Wright monetary equilibrium. Suppose agents believe that Π=1, Θ=1, and Ψ=0. The payoffs to money and credit trading are then given below.

$$(r + \beta x(g+m))(V_m-V_g) = \beta gx(1-x)(u-\epsilon+V_n-V_g) > 0 \tag{1.14}$$

$$(r+\alpha)(V_l-V_g) = [\ \alpha - \beta gx^2(\frac{r+\beta gx+\beta m}{r+\beta gx+\beta mx})\ ](u-\epsilon+V_n-V_g) \tag{1.15}$$

$$(r+\alpha)(u-\epsilon+V_d-V_n) = r(u-\epsilon) + \alpha(u-\epsilon+V_n-V_g) > 0 \tag{1.16}$$

When it is believed that money but not credit will be used in trading, then the return to accepting money is unambiguously positive and all agents will set $\pi=1$. Under these beliefs there is also a positive payoff to borrowing goods. Thus, to ensure no credit arises, the return to lending goods must be negative ($V_l<V_g$) in equilibrium. If the arrival rate of production opportunities is small enough then this return will indeed be negative. In particular, given $u-\varepsilon+V_n-V_g>0$, if

$$\alpha \;<\; \beta g x^2\left[\frac{r+\beta g x+\beta m}{r+\beta g x+\beta m x}\right] \tag{1.17}$$

then no one will lend in equilibrium. Since $x\leq 1$, the right hand side of equation (1.17) is greater than $\beta g x^2$. The restriction is therefore less strict than the barter equilibrium condition.

Therefore, if agents believe that other agents set $\Pi=1$, $\Theta=1$, and $\Psi=0$ (ie. agents are willing to borrow but not lend) then each agent will set his own trading strategies to be the same as long as the parameter restriction in equation (1.17) holds. We see here again that beliefs are sufficient to support the use of money but an extra condition on the arrival rates is required to ensure credit does not emerge.

1.5 Barter/Credit Equilibrium:

In this section we shall consider an equilibrium in which barter and credit, but not money, are used in trade. If an agent believes that all other agents will set $\Pi=0$, $\Theta=1$, and $\Psi=1$ then a barter and credit equilibrium may result.

$$r(V_m - V_g) = -\beta g x^2(u - \epsilon + V_n - V_g) - \beta n x^2(V_l - V_g) \tag{1.18}$$

$$(r + \alpha + \beta n x^2)(V_l - V_g) = (\alpha - \beta g x^2)(u - \epsilon + V_n - V_g) \tag{1.19}$$

$$(r + \alpha + \beta g x^2)(u - \epsilon + V_d - V_n) = r(u - \epsilon) + \alpha(u - \epsilon + V_n - V_g) > 0 \tag{1.20}$$

The return to borrowing given in equation (1.20) is still positive for all parameters. If the arrival rate of production opportunities is larger than the arrival rate of double coincidence of wants trading partners, ie. $\alpha > \beta g x^2$, then there will be a positive return to lending ($V_l - V_g > 0$) and credit will arise in equilibrium. With agents believing no one else will accept money in trade and $V_l - V_g > 0$, the return for an individual agent to accept money will be negative, ie. $V_m - V_g < 0$, so agents will set $\pi = 0$.

Therefore, when agents believe that trading strategies will be $\Pi = 0$, $\Theta = 1$, and $\Psi = 1$, they will find it optimal to follow the same strategies provided $\alpha > \beta g x^2$.

1.6 Barter/Credit/Money Equilibrium:

In this section, we shall consider the full equilibrium in which agents always trade for money and always use credit when possible. Thus each agent believes that other agents are setting $\Pi = \Psi = \Theta = 1$ and that it is in their interest to do the same.

Equations (1.1) to (1.5) are once again used to derive expressions for the payoffs to money and credit trades. Equations (1.21), (1.22), and (1.23) show the returns to accepting money, lending, and borrowing, respectively.

$$P_1(V_m-V_g) = \beta x \{ g(1-x) - \frac{nx}{P_2}[\alpha-\beta gx^2(\frac{r+\beta gx+\beta m}{P_1})] \} (u-\epsilon+V_n-V_g) \quad (1.21)$$

$$P_2(V_l-V_g) = [\alpha - \beta gx^2(\frac{r+\beta gx+\beta m}{P_1})] (u-\epsilon+V_n-V_g) \quad (1.22)$$

$$(r+\alpha+\beta gx^2)(u-\epsilon+V_d-V_n) = r(u-\epsilon) + \alpha(u-\epsilon+V_n-V_g) > 0$$

$$\textit{where: } P_1 = r+\beta gx+\beta mx \quad (1.23)$$

$$P_2 = r+\alpha+\beta nx^2(\frac{r+\beta gx}{P_1})$$

Given $u-\epsilon+V_n-V_g>0$, if the arrival rate of production opportunities is sufficiently large, then equation (1.22) shows that the return to lending, V_l-V_g, will be positive. In particular, we require that

$$\alpha > \beta gx^2 [\frac{r+\beta gx+\beta m}{r+\beta gx+\beta mx}] \quad (1.24)$$

Therefore, the arrival rate of production goods must be sufficiently larger than the arrival rate of a double coincidence of wants trading partner. Although this condition is the opposite of equation (1.17), that governing the existence of a barter and money equilibrium, it does not preclude the possibility that the two equilibria could coexist in the same parameter space since the value of the population fraction g is different in the two equations. Given $V_l-V_g>0$ and the peoples belief that $\Pi=1$, then there will be a positive return to trading for money only if the following condition is satisfied.

$$g(1-x)\frac{P_2}{nx} > \alpha-\beta gx^2[\frac{r+\beta gx+\beta m}{r+\beta gx+\beta mx}] \quad (1.25)$$

One manner in which this can be interpreted is that the arrival rate of production goods or loan repayments cannot be too much larger than the arrival rate of double coincidence of wants trades. Therefore, credit trades must occur faster than barter trades but not too much faster.

Therefore, if agents believe that $\Pi=\Theta=\Psi=1$ and the conditions in equations (1.24) and (1.25) hold, then there will be an equilibrium in which each agent sets his own strategies to be the same and both money and credit are used in trades. Although having both money and credit being used in trades will increase the potential number of trades, it may not necessarily be welfare improving as will be shown later.

Equations (1.6) to (1.10) can be solved to find expressions for the population fractions. The number of goods traders in steady state will be the real root of the cubic equation (1.26). The number of non-debtor producers and lenders can then be found from equations (1.27) and (1.28), respectively.

$$2\beta^2 x^4 g^3 + 2\beta^2 m x^3 g^2 + [\alpha\beta x^2 + \alpha\beta m x(1-x) + \alpha^2] g - \alpha^2(1-m) = 0 \tag{1.26}$$

$$n = \frac{\beta g x(gx + m)}{\alpha - \beta g x^2} \tag{1.27}$$

$$l = d = \frac{\beta^2 g^2 x^3 (gx + m)}{\alpha(\alpha - \beta g x^2)} \tag{1.28}$$

Consider now the usefulness of money and credit in this equilibrium. Following along the lines of Kiyotaki and Wright (1989), we shall examine the number of agents, the number of trades, velocity, and acceptability of barter, money, and credit. The number of agents with goods, money, or credit (g, m, or l, respectively) can vary widely depending on the parameter values chosen. Total differentiation of equation (1.26) shows that $\partial g/\partial m<0$ and $\partial g/\partial x<0$. The signs of the derivatives of l with respect to m and x are less certain but they to seem to be negative for most parameter values.

The parameter β is the arrival rate of trading partners or the expected number of trading partners to arrive in a specified time interval, say ω. The expected number (or arrival rate) of barter trades in the same interval ω is then βg^2x^2. The number of credit trades is $\beta gnx^2\theta\psi$ while $\beta gmx\pi$ is the number of money trades. A similar measure is the number of times a commodity is offered for trade. This number will, of course, be larger than the number of times a commodity is actually traded. The number of times a good is offered for barter trade in the given period is βg^2x. The number of offers of money and credit are βgmx and $\beta gnx^2\psi$, respectively.

Velocity is then defined as the number of trades using a certain commodity divided by the number of agents with that commodity. The velocity of barter is thus βgx^2 while the velocity of money is $\beta gx\pi$. The velocity of credit is $\beta gnx^2\theta\psi/l$. Since $0<x<1$, we know the veloci.y of money in the full equilibrium is greater than that of barter. By equation (1.10), the velocity of credit is constant at the arrival rate of production opportunities, α. Using the restriction in equation (1.24), the velocity of credit is greater than that of barter but may be less than or greater than the velocity of money. Since $\partial g/\partial m<0$, the velocities of both money and barter fall as the money supply increases and, in the limit, equal zero when $m=1$. Hence, for large money supplies, the velocity of credit will eventually be larger than the velocities of both money and barter.

From equations (1.2) and (1.5), the differential between the value of lending (V_l) and value of money (V_m) can be computed.

$$(r + \beta g x \pi)(V_l - V_m) = (\alpha - \beta g x \pi)(u - \epsilon + V_n - V_l) \tag{1.29}$$

The first term on the right hand side is, of course, the differential between the velocities of money and credit while the second term, which will be positive in the full equilibrium, is the lender's dividend and capital gain from having the loan repaid. Therefore, the value function will be larger for that instrument (money or credit) which has the greater velocity of circulation.

A commodity's acceptability is defined as the number of trades in which it is used divided by the number of times it is offered for trade. The acceptability of barter is thus equal to x while the acceptability of credit is $x\theta$. The acceptability of money is π. Barter and money are thus acceptable for only a fraction of the time in the full equilibrium since it is more difficult to satisfy the double coincidence of wants. Money is fully acceptable, $\pi=1$, since everyone values money and will trade for it. Note that acceptability does not respond to changes in the money supply.

The creation of this equilibrium was one of the primary purposes of this chapter. It is an equilibrium in which money and credit coexist. Since an agent never has simultaneous access to both money and credit, neither mechanism is able to drive out the other. An agent may prefer using one or the other but both are useful in lessening the trading frictions and so are used when the opportunity arises.

1.7 Examples:

The model can easily be simulated on a computer once some values for the exogenous parameters are assumed. The values of α=0.1, β=1, r=0.1, u=100, and ε=0.1 were chosen since they ensure the existence conditions for all four equilibria are satisfied. Figure 1.1 gives the area for which the full equilibrium and the barter/money equilibrium exist in (x,M) space. Along the horizontal axis, with M=0, the barter equilibrium exists for the interval x=(0.4472, 1) and the barter/credit equilibrium for the interval x=(0,1).

If agents beliefs are consistent with a barter equilibrium then x=0.4472 implies V_l-V_g=0. Above this x value the lending return is negative while below it lending yields a positive return even though agents believe no one else will lend. When x becomes too small, the parameter restriction $\alpha<\beta gx^2$ is violated and the barter equilibrium dissolves as agents start to lend.

From Figure 1.1 we can see that the full equilibrium (area ABCD) ceases to exist if the x value becomes too high for any given money supply. This occurs because, for these high x values, V_m-V_g<0. The higher is x the more effective is barter and therefore the less necessary is money.

The barter/money equilibrium (area EFG) exists only for the southeast corner of the (x, M) parameter space. Outside that area it becomes profitable to lend (V_l-V_g>0) even while believing no one else will lend. As x falls for a given money supply, double coincidence of wants trades become less frequent and hence the benefit of credit trades increases until finally they are profitable. For higher M, less of a fall in x is required because a greater money supply implies a lower g and therefore a greater possibility that the parameter restriction in equation (1.17) (V_1-V_g<0) is violated.

From Table 1.1 we can see that for low values of x, under approximately 0.4, the full equilibrium with barter, money, and credit being used in trades will provide a higher level of welfare than the equilibrium with simply barter and credit.[3] Also, full equilibrium welfare is increasing in the money supply but only for small money balances and low x values. Money may thus be welfare improving, as in Kiyotaki and Wright (1990), even in the presence of credit. As the number of trading partners increases, however, the usefulness of money declines so that, above x=0.4, the barter/credit equilibrium provides a higher welfare level. In this example, for x values between about 0.4 and 0.6, the barter/credit equilibrium provides the greatest utility among the four pure strategy equilibria. For x=0.6 and above, an equilibrium with barter and money will yield a higher utility level than will barter and credit. However, barter alone will provide an even greater welfare level. No example was found in which the barter/money equilibrium yielded the greatest welfare of the four equilibria. For lower x values, it was dominated by equilibria with credit and, for higher x values, by the barter equilibrium.

Again note from Table 1.1 that money may be welfare improving as it was in the Kiyotaki and Wright (1990) models. For x values from 0 to approximately 0.35, welfare in the full equilibrium initially increased with the money supply before declining thereafter. The same was true for the barter and money equilibrium for x values of around 0.45. However, for higher values of x the optimal money supply was zero.

---

[3] The welfare level is a weighted average of the various value functions using the population fractions as weights.

An idea of the model's comparative statics can be found by varying the parameters in these numerical simulations. If the arrival rate of production opportunities (α) increases then the area covered in (x,M) space by the full equilibrium expands to the northeast. In contrast, the areas of the barter/money and barter equilibria shrink as α rises. The barter/credit equilibrium still exists for all x values. These effects are not surprising since an increase in α makes credit more attractive by raising the rate of payoff and ensures equation (1.24) is more easily satisfied (existence of full equilibrium condition) and equation (1.17) is more difficult to satisfy (existence of barter/money equilibrium condition). The increase in α also raised the welfare levels in all four equilibria. This is again as expected since production is made easier and hence there will be a greater number of agents with goods.

As could be expected, given the form of the parameter restrictions governing existence, an increase in β generally had the opposite effect of an increase in α. The area of the full equilibrium shrunk to the southwest as β rose while the barter and barter/money equilibria areas expanded. There appeared to be no change in the barter/credit equilibrium. As trading partners were now arriving faster in all four equilibria, welfare levels were raised in each. However, welfare did not increase as quickly nor as substantially as for changes in α.

An increase in the subjective rate of time preference (r) expanded the area of the full equilibrium. The barter/money equilibrium area decreased to the southeast in (x,M) space. The areas of the barter/credit and barter equilibria were unaffected by the change in r since r did not enter the parameter restrictions imposed for these equilibria. The welfare levels were negatively related to r for all four equilibria.

The final two parameters, u and ε, did not affect the areas covered by the equilibria. Welfare levels did, however, change in a predictable manner by varying positively with net utility, u-ε.

1.8 Credit Rationing Equilibrium:

This chapter has considered only pure strategy equilibria so far despite there being multiple mixed strategy equilibria as well. One particular mixed strategy could be considered to be a credit rationing equilibrium. Suppose agents believe that other agents are willing to borrow goods ($\Theta=1$) but that they will lend for only a fraction of the opportunities which arise ($0<\Psi<1$). The steady state equilibrium that results from these beliefs will have certain agents who are successful borrowers as well as other, identical, agents who wish to borrow but who are refused credit (ie. rationed) by the lenders (goods traders). Such an equilibrium would exist both with money and without money. Below is a description of the equilibrium with money.

Equations (1.30) to (1.32) below represent the dividend and capital gain returns from trading for money, lending, and borrowing, respectively.

$$P_1(V_m - V_g) = \beta g x(1-x)\,(u-\epsilon+V_n-V_g) > 0 \tag{1.30}$$

$$P_2(V_l - V_g) = [\,\alpha - \beta g x^2(\frac{r+\beta g x+\beta m}{P_1})]\,(u-\epsilon+V_n-V_g) = 0 \tag{1.31}$$

Equation (1.32) shows that the return to borrowing will again be unambiguously

$$
\begin{aligned}
(r+\alpha+\beta g x^2 \Psi)(u-\epsilon+V_d-V_n) &= r(u-\epsilon) + \alpha(u-\epsilon+V_n-V_g) > 0 \\
\textit{where:}\quad P_1 &= r+\beta g x+\beta m x \\
P_2 &= r+\alpha+\beta n x^2 \Psi\left(1+\frac{\beta m x}{P_1}\right)
\end{aligned}
\tag{1.32}
$$

positive. There will be a mixed lending strategy ($0<\psi<1$) if the lending return, V_l - V_g, is zero. This is ensured by assuming the condition in equation (1.33) where g is shown as a function of ψ to emphasize the mechanism through which this lending strategy affects the lending return. When this is imposed on equation (1.30), the retur to accepting money will also be positive. If agents believe that others will play a mixed lending strategy, $\Psi<1$, that will satisfy equation (1.33), then there will be a steady state credit rationing equilibrium when they set their own ψ to be the same.

$$
\alpha = \beta g(\psi) x^2 \left[\frac{r+\beta g(\psi)x+\beta m}{r+\beta g(\psi)x+\beta m x}\right]
\tag{1.33}
$$

Figure 1.2 shows the area of the credit rationing equilibrium with money in the (x,M) space using the same parameters as specified for the simulations above. The equilibrium exists in the same area as the barter/money equilibrium. This is not surprising of course since the boundary of each equilibrium specifies V_l-V_g=0 for ψ=0. In the interior of the credit rationing equilibrium, however, V_l-V_g is still zero, instead of negative, and the probability of lending, ψ, is positive instead of zero. As you move to the southeast in (x,M) space, the value of ψ rises. For instance, at x=.7 and M=0.2555, if agents believe that other agents will only lend 10% of the time then V_l-V_g=0 and a steady state equilibrium will result if each agent sets the same strategy. However, at x=.95 and M=.0228, agents must believe that others will lend 65% of the time for there to be an equilibrium with a mixed lending strategy. The maximum mixed lending strategy was just above .7 for this example.

Credit rationing models have generally been based on either loan market imperfections or, more recently, imperfect information. In the imperfect information models, banks are assumed to know only the average or expected risk or return levels of projects and firms. As a result they cannot properly distinguish between borrowers. Jaffee and Stiglitz (1990) describes adverse selection and incentive effects that can cause an inverse relationship between interest rates and lender return for higher interest rate levels. When this occurs, banks will not lend to any borrowers at rates above the critical interest rate that maximizes the banks' return because this would lower their return. Borrowers willing to pay more than this critical rate would be refused credit leaving an unfulfilled excess demand in equilibrium.

In this model, however, credit rationing is not the result of any imperfect information problem. The lenders are perfectly informed as to all of the borrowers' characteristics. Borrowers are credit rationed as the result of a self-fulfilling belief. Agents believe there will only be a thin credit market and, hence, that is what occurs in the steady state equilibrium. In the area for which the full equilibrium and credit rationing equilibrium coexist, if the agents could simply raise their beliefs about the use of credit, then rationing would disappear.

This credit rationing equilibrium is, however, not robust to the introduction of negotiated probabilities for the delivery or repayment of the goods. Suppose, in the manner of Diamond (1990), there is a probability p for the delivery of the lender's good and a probability q for the repayment of the debtor's good that are determined optimally through negotiation. There will always be some p and q in the unit interval that w... give both the lender and borrower positive payoffs. Since the probabilities are set by the agents, they will not choose so that the return to lending is zero. There would thus no longer be a credit rationing equilibrium.

1.9 Conclusions:

This chapter has shown that it is possible to generate a search model in which agents will trade using both money and credit in equilibrium. The introduction of credit has opened up more possibilities for trade among the agents and thus there is a potential, although uncertain, that the agents may be better off with credit instead of, or in addition to, money. For the lower values of x for which the equilibria existed, the agents were better off using credit.

Beliefs about the acceptability of money will determine whether or not money is used in trade. In contrast, beliefs about whether other agents want to borrow and lend are not enough to ensure that there are, or are not, credit trades in equilibrium. Further parameter restrictions are required to guarantee that the payoffs are consistent with beliefs. In particular, in order for there to be credit the arrival rate of production opportunities must be at least as large as the arrival rate of trading partners forming a double coincidence of wants.

The analysis in this chapter has focused solely on agents who have known the production good of their trading partners. If, however, an agent's production good was private information or even if it was a random draw from the production set, equilibria with credit would still exist. Lenders would face a probability x of being repaid a consumable good and a probability 1-x that the good that is produced by the debtor is non-consumable and a capital loss is incurred. If x and the utility from consumption are large enough then credit may arise. As with the equilibria discussed in the sections above, a condition requiring a sufficiently large arrival rate of production opportunities must be satisfied.

The model could be further expanded to allow for the borrowing and lending of money as well. Since agents in debt must repay their loan immediately upon production, it is conceivable that an agent could borrow money and produce while in debt. The inventory assumption restricting holdings to only one asset type would not be violated because the debtor would never really hold both a good and money. The newly produced good would be given to the lender immediately upon completion. Agents would want to lend money if production opportunities arrive fast enough and agents would borrow money because it would give them something that could be traded for a good while they were still in the production process.

Further work is also possible with the model to respond to criticisms of some of the model's basic assumptions. For instance, the model could, in some way, be expanded to allow for perfectly divisible goods or for the holding of more than one type of asset. The model could then be used to consider questions regarding prices, inflation, and monetary policy in general. To date, economists have yet to devise a satisfactory model that relaxes these assumptions.

Much more work is possible with this type of model in terms of multiple asset holdings and divisible assets. However, there is also still much that can be learned from the simpler models derived to date.

Appendix 1.1

Proposition 1: $u-\varepsilon+V_n-V_g>0$

Proof: This shall be a proof by contradiction. Suppose that $u-\varepsilon+V_n-V_g<0$. Then there will be no goods-for-goods trades in this economy. Consider then the four possible cases.

Case 1: $V_m-V_g>0$, $V_l-V_g>0$

Trading for money and lending goods is still profitable. We then have that $u-\varepsilon+V_n-V_l<0$, implying that potential return while in the lending state would be negative and thus that agents would choose not to lend. Therefore, $V_l=0$. This implies that V_m and V_g are both zero, a contradiction.

Case 2: $V_m-V_g>0$, $V_l-V_g<0$

We then have that $u-\varepsilon+V_n-V_m<0$ and $V_l-V_g<0$ so no one will want money and $V_m=0$. Therefore $V_g=0$, a contradiction.

Case 3: $V_m-V_g<0$, $V_l-V_g>0$

The only option for goods traders is to lend but with the supposition and $V_l>V_g$ the return while in the lending state is negative. Hence, we once again have $V_g=0$, a contradiction.

Case 4: $V_m-V_g>0$, $V_l-V_g>0$

With all possible trades yielding negative payoffs, we get $V_g=0$, a contradiction.

Therefore, this economy shall have $u-\varepsilon+V_n-V_g>0$.

Table 1.1: Welfare Levels

X	Barter	Barter/Money M=0.001	Barter/Money M=0.01	Barter/Credit	Full Equil. M=0.001	Full Equil. M=0.01
0.05	-	-	-	2.490	2.534	2.919
0.3	-	-	-	42.737	42.756	42.923
0.35	-	-	-	47.286	47.295	47.359
0.4	-	-	-	50.742	50.742	50.731
0.45	50.157	50.164	50.223	53.401	53.394	53.320
0.5	53.614	53.614	53.606	55.471	55.458	55.336
0.55	56.648	56.614	56.575	57.113	57.095	56.933
0.6	59.326	59.313	59.195	58.429	58.408	58.212
0.65	61.703	61.684	61.520	59.499	59.474	59.250

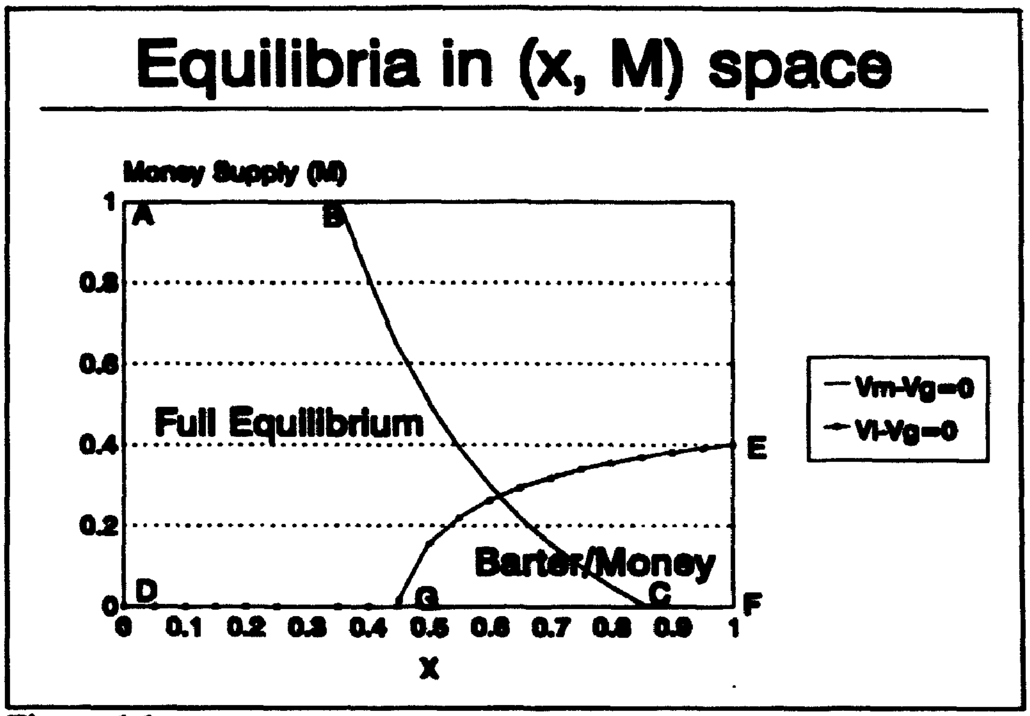
Equilibria in (x, M) space
Money Supply (M)
1
0.8
0.6
0.4
0.2
0
A
B
Full Equilibrium
E
Barter/Money
D
G
C
F
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
X
Vm-Vg=0
Vl-Vg=0

Figure 1.1

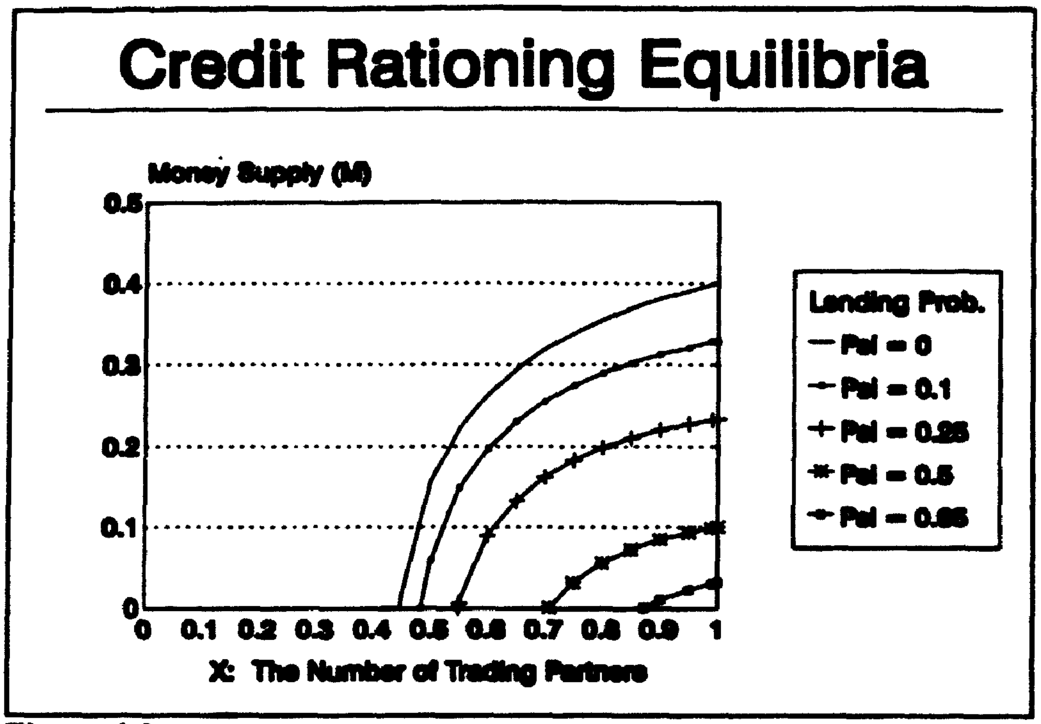
Credit Rationing Equilibria
Money Supply (M)
0.5
0.4
0.3
0.2
0.1
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
X: The Number of Trading Partners
Lending Prob.
Psi = 0
Psi = 0.1
Psi = 0.25
Psi = 0.5
Psi = 0.65

Figure 1.2

CHAPTER 2: PRICES IN A SEARCH MODEL OF MONEY

2.1 Introduction:

One of the criticisms of search models of money (such as Kiyotaki and Wright, 1990) is that there are no prices explicitly determined in the model. Goods and money are always traded on a one-for-one basis without any bargaining mechanism to determine nominal prices. The purpose of this chapter will be to introduce a price determination mechanism into the model by relaxing some of the more restrictive assumptions. The model will partially relax the inventory restrictions on money of the typical search model by allowing agents to store more than one unit of money but still only one unit of goods. A Nash bargaining game is chosen as the mechanism through which agents determine the amount of money and goods to be traded in any given match. The neutrality of money in the model can be studied through this introduction of prices.

Previous search models of money, in particular Kiyotaki and Wright (1990 and 1991), have assumed that each agent could hold only one real unit of money. It was sometimes argued that money must therefore be neutral since any change in the nominal money supply would immediately be accompanied by the same percentage change in prices so that the real money supply, and the solution to the model, was left unchanged.

The model, however, cannot explain how this change in prices comes about. It is simply assumed to occur somewhere in the background. There is no change in the solution to these search models if money traders are interpreted as holding one nominal unit of money instead of one real unit. The immediate implication, however, is that money is no longer neutral since nominal prices are always fixed at one for every level of the money supply. If agents can only hold a restricted amount of money or goods in inventory, then the number of units actually held in equilibrium will matter for the determination of prices and money cannot be neutral in the model.

For a given level of nominal money supply, Kiyotaki and Wright have calculated a price level for special types of equilibria in which money traders and goods traders receive the same surplus from trading. In their model, the real money supply is also the number of agents holding a real unit of money. If this real money supply is endogenously selected to find an equilibrium that evenly divides the available surplus between the agents involved, then a price level can be calculated for a given nominal money supply. This price level, however, is basically a residual and the model cannot explain the bargaining process through which it is determined.

This chapter will permit agents to hold multiple units of money and derives a model with explicit price determination as the result of bargaining between agents. This permits the analysis of money neutrality in the context of search models of money and will illustrate the importance of the inventory restrictions on money. In the limit, as all restrictions are removed from money inventory levels, money is completely neutral in the model. However, with money inventory restrictions still in effect, although less strictly than in previous models, money will be non-neutral. The extent of the deviation from money neutrality is examined.

Shouyong Shi (1993) also examines the determination of prices within a search model of money. His paper takes the opposite approach to the model proposed here by introducing a divisible consumption good but still having money stored in one unit quantities. Shi's model yields two monetary equilibria. The purchasing power of money exhibited different and often opposite features in the two equilibria following changes in the model's parameters. There was nothing to determine which of the two equilibria would actually occur. The model also has the added feature of a completely divisible price but it essentially considers the same problem as this chapter.

Kiyotaki and Wright also examined the taxation of money in the context of their 1990 model. They found that these inflation taxes did tend to increase the price level in their split-the-surplus equilibrium and were also able to generate a Laffer curve for government tax revenue. Inflation taxes in the context of the model in this chapter also increase prices but do not yield a Laffer curve in government revenue. Tax revenue tends to increase monotonically with the tax rate.

The extensions proposed in this chapter expand the applicability of the model but also greatly complicate the calculations so that numerical solutions figure prominently in the analysis. Section 2.2 below outlines the nature of the model when money inventory restrictions have been relaxed but not completely removed. Section 2.3, on the other hand, describes the format of the model if money inventories can be any non-negative real number. Section 2.4 gives the results of some of the simulations. Section 2.5 studies the effect of an inflation tax on prices and Section 2.6 investigates the comparative statics of the model. A conclusion is provided in the final section.

2.2 Model:

The basic model is similar to Kiyotaki and Wright (1990). There is a continuum of agents with unit mass, uniformly distributed on a circle of circumference two. An agent's type is represented by the good which that agent most prefers to consume. For simplicity, agents receive a fixed utility of u, net of any costs, from consuming any good within an exogenous distance $x \in (0,1]$ of their most preferred good or agent type. The probability that a randomly selected agent has a good which will yield utility is thus x and the probability of a double coincidence of wants is x^2. Assume that agents cannot consume their own production good and that, like agent types, production goods are uniformly distributed on the circle of circumference two. Assume that no technology exists which will permit credit to emerge.

Kiyotaki and Wright (1990) assumed that agents can only hold one type of commodity (a good or money) at any given time and may only hold one real unit of that commodity. This chapter shall relax these two assumptions to a certain extent and then examine the determination of prices. First, assume that agents can simultaneously hold both money and goods. Second, assume that agents can still store only **one** unit of the good but may store up to M discrete nominal units of money, where $M \in \{1,2,3,...\}$. Having agents holding nominal money instead of real money is a slight variation from the Kiyotaki and Wright model and it is primarily a difference in interpretation only. The main innovation in this model is in the storage technology, ie. having agents storing multiple units of both money and goods.

Goods inventories are still restricted to a single unit and the value of M kept relatively small in order to keep the model manageable. Ideally, M should be set quite high in order to achieve a fine enough grid so that money might become neutral. However, large values of M are quite computationally intensive and will be left for subsequent work.

Since production would not add substantially to the model, it is assumed that production is instantaneous and costless so that each agent will always have a unit of his production good in inventory. There are then M+1 possible states of nature for an agent where the state is identified by the number of units of money that are held. The total nominal money supply is given by $S=\Sigma^{M}_{i=0} m\pi(m)$ where $\pi(m)$ represents the fraction of agents with m units of money. Assume that the money supply S is initially distributed in some manner across agents. The distribution of money across agents will move towards a steady state distribution once the economy begins operating. The total supply of money is restricted above by M if each agent happens to hold the maximum permitted inventory. A unit of money is interpreted as the smallest denomination into which the currency can be divided.

Assume that no indirect barter will occur. Such a result can be justified if the costs of producing another unit of the good are lower than the transaction costs of accepting an unconsumable good through indirect barter. For simplicity, such a transaction cost is not modeled here.

A monetary equilibrium in this model requires that agents believe in the value of money and believe that other agents will also accept money. A monetary equilibrium is then given by a value function, V(m), a population density function, $\pi(m)$, and a series of optimally chosen prices that support the beliefs of the agents.

The function V(m), represented by equation (2.1) in Appendix 2.1, gives the value function for a typical agent holding m units of money, for $m \in \{0,1,2,\ldots,M\}$. In order to trade for their consumption goods, agents must search for trading partners who are located according to an exogenous, fixed, Poisson arrival rate of $\beta>0$. With probability $\pi(m')$ the arriving partner has m' units of money in inventory. There is a double coincidence of wants between the trading partners with probability x^2. The agents trade, consume the good for a utility of u, produce a new good, and then move into a state with $\tilde{m}$ units of money in inventory. With probability x(1-x), there is a single coincidence of wants in which only the agent in question desires the other's good. After buying the other agent's good with money, the agent will consume and emerge in a state with $\tilde{m}_1<m$ units of money in inventory. With the same probability (1-x)x, the single coincidence of wants is reversed. In this case, the agent doesn't consume but is able to sell his good in exchange for money and enters into a new state with $\tilde{m}_2>m$ units of money in inventory.

To determine the precise level of payoffs for a particular match, an assumption with respect to the bargaining process is required. Assume that, when two agents meet, a Nash bargaining game is played in order to select that price level (ie. the amount of goods and money to be traded) which maximizes the joint surplus of the two agents. The price level must, of course, be chosen from among the feasible set of prices defined by the inventories of the agents. The maximization problems are written with agents choosing their optimal money inventory levels with which they emerge from trade. The problem could also have been set up in terms of the price levels themselves but the format chosen seemed to keep the presentation simpler.

During a double coincidence of wants trade, the agents will participate in a bargaining game to determine whether any money is to be exchanged in addition to the one-for-one barter trade of goods. The agents will select their final money holdings, $\tilde{m}$, which may be larger or smaller than their initial holdings. The more dissimilar were the two agents, the more likely it was that one agent had to pay some money in addition to his good in order to acquire the other agent's good. The Nash bargaining game for a double coincidence of wants is represented by the function $\tilde{m}(m,m')$ in Appendix 2.1.

With probability x(1-x) there is a single coincidence of wants match in which the agent in question desires his trading partner's good. The agent pays $(\mathbf{m} - \tilde{\mathbf{m}}_1)$ units of money to the trading partner in exchange for the other's good, which is immediately consumed. No agent will accept a good in trade that cannot immediately be consumed. The maximization problem, $\tilde{\mathbf{m}}_1(\mathbf{m},\mathbf{m}')$, for this type of match is given by equation (2.3).

With probability (1-x)x, the single coincidence of wants problem is reversed and the agent sells his good for money. After selling the good, a new unit is immediately produced and the agent emerges from trade with $\tilde{\mathbf{m}}_2$ units of money. This maximization problem is given by equation (2.4) or $\tilde{\mathbf{m}}_2(\mathbf{m},\mathbf{m}')$ in Appendix 2.1. This problem is of course simply the mirror image of $\tilde{\mathbf{m}}_1(\mathbf{m},\mathbf{m}')$.

The bargaining process selects the integer level of the price level that will maximize the product of the payoffs of the two agents. An aggregate price level, P, can be calculated as the weighted average of these prices which are calculated for individual matches. The weights used are the frequencies of the particular matches. Let P_m represent the average number of units of money paid for a good in single coincidence of wants trades and P_{gm} the average amount of money that is traded in addition to the good in double coincidence of wants trades. The sum $P+P_{gm}$ represents the average total nominal value paid by an agent to acquire a good in a double coincidence of wants trade. The value of P_{gm} was often but not always zero. An overall aggregate price level, P, is calculated as

$$
\begin{aligned}
P &= \frac{A*P_m + B*(P + P_{gm})}{A + B} \\
&= P_m + \frac{B}{A}*P_{gm}
\end{aligned}
$$

where A and B are the number of single and double coincidence of wants trades, respectively. Following an increase in S, P may increase as a result of two factors: i) the negotiated price levels for particular matches may rise and ii) the steady state population distribution of agents may change and shift agents toward matches with higher prices.

Note that the negotiated price in each match in the model is a function of the money inventory levels of the trading partners directly involved and of whether there is a single or a double coincidence of wants between the agents. The price a particular agent receives for his good will change with each match even though his production good is always the same. Prices are determined for each match individually and there is no mechanism by which agents can arbitrage to ensure that one price will prevail throughout the economy for a given good. The price for a good will vary depending on the characteristics of the agents involved in each particular trade. There will be no law of one price which holds in this model.

The model in Shi (1993) assumes agents can store either one production opportunity (not actual goods but just an opportunity to produce) or one unit of money. As such, matches between goods traders and money traders always involve agents with the same assets or inventories and hence yield the same price. That model generates a general price level that prevails throughout every match in the whole economy. The model here, however, has a different price for each match involving agents with different assets in inventory.

The model is closed by specifying in equation (2.5) the laws of motion governing the population fractions of agents, $\pi(m)$. In steady state, the flow into the state holding $m=a\geq 0$ units of inventory must equal the flow out of that state.

Before simulating this model, the next section will describe the nature of the model without any restrictions of money holdings.

2.3 The Model Without Monetary Inventory Restrictions:

Although it proved intractable to solve the model without restrictions on inventory holdings of money, it was possible to set up the model for this case and show that the neutrality of money would hold. Assume now that there is no restriction on the maximum money inventory holdings of an agent, ie. $M=\infty$, but that goods inventories are still constrained to be either zero or one unit. Let the inventory holdings of money be completely divisible so that $m \in [0,\infty)$. Appendix 2.2 contains the detailed equations describing an equilibrium of this unrestricted model.

An equilibrium is defined by a value function, equation (2.6), a series of price functions, equations (2.7) to (2.9), and a set of equations defining the population density function across inventory holdings, equation (2.10). When agents believe that money will be used as a medium of exchange and this set of equations is satisfied then an equilibrium will exist.

The value function in equation (2.6) is quite similar to that in equation (2.1) except that now you must integrate across all possible trading partners' inventories $m' \geq 0$. The Nash bargaining problems for the prices, equations (2.7) to (2.9) have the same specification as before. Equation (2.10) shows the steady state equation for the number of agents with $m \in [0,a]$ for all $a \geq 0$. The positive terms represent the leftward flow of agents at a into $m \in [0,a]$ for all $a \geq 0$. The negative terms represent the rightward flow out of $[0,a]$. When these flows are balanced there will be a steady state for the population distribution of agents.

Proposition 1 shows that the neutrality of money may hold in this model when there are no restrictions of money inventories. The proposition asserts the existence of an equilibrium at the new, higher money supply which exhibits money neutrality (ie. no real variable changes from the initial equilibrium). However, this is a steady state not a dynamic model so the proposition does not guarantee that this particular equilibrium will be the one that emerges should there be multiple equilibria at the new and higher supply of money.

__Proposition 1:__ Let V(m), π(m), $\tilde{m}$(m,m'), $\tilde{m}_1$(m,m'), and $\tilde{m}_2$(m,m') describe an equilibrium satisfying equations (2.6) to (2.10) in Appendix 2.2. Now define Y(m), θ(m), $\bar{m}$(m,m'), $\bar{m}_1$(m,m'), and $\bar{m}_2$(m,m') for all m,m',a≥0 as follows:

$$Y(m) = V(1/2m)$$

$$\int_0^{2a} \theta(m)\,dm = \int_0^{a} \pi(m)\,dm \quad \forall a \geq 0, \qquad \frac{1}{2}\int_0^{\infty} \theta(m) m\,dm = \int_0^{\infty} \pi(m) m\,dm = S_0$$

$$\bar{m}(2m,2m') = 2\tilde{m}(m,m'), \quad \bar{m}_1(2m,2m') = 2\tilde{m}_1(m,m'), \quad \bar{m}_2(2m,2m') = 2\tilde{m}_2(m,m')$$

Then this new set of functions will also define a steady state equilibrium.

The basic idea of the proof, given in Appendix 2.2, is to show that Y(m), θ(m), $\bar{m}$(m,m'), $\bar{m}_1$(m,m'), and $\bar{m}_2$(m,m') also satisfy the equilibrium conditions in equations (2.6) to (2.10). The proposition shows the neutrality of money by showing there will exist a new equilibrium, with double the money supply, in which no real variable changes. After the increase in the aggregate money supply from S_0 to $2S_0$, agents can receive the same welfare for holding twice as much money as they previously held, Y(2m)=V(m) for all m≥0. For any given positive real number *a*, there will exist an equilibrium in which there will be the same number of agents with inventories of less than *2a*, after the doubling of the money supply, as there were with less than *a* before the increase. The distribution of money holdings can thus remain unchanged. Finally, all price levels can double so that agents entering a trade with twice as much money as before will also exit the trade with twice as much as they would have with the lower money supply, ie. $2\tilde{m}(m,m') = \overline{m}(2m,2m')$. Money in this model is thus neutral if there are no restrictions on the agents' inventories of money.

It was not possible to find either an analytical or a numerical solution to the model in this section. Consequently, the next section of the chapter relates the results from simulating the model with finite maximum money holdings. Those simulations however show that the model is far from being neutral when there are inventory restrictions as in many search models of money that are currently being used. In particular, prices will not double on an individual or aggregate level following a doubling of the money supply.

2.4 Simulations:

This section will record the results of some of the simulations that were run with the model when money inventories were restricted to finite values. The following baseline parameters were assumed throughout: β=5, r=.1, and u=100. Equilibria shall be found for maximum money inventory holdings of M = 1, 2, 4, and 8. Given the choices open to the agents there may be a number of possible equilibria which exist for the same parameter values.

Solving for equilibria involved specifying an initial strategy set and then calculating the associated population fractions from steady state equations (2.5) derived from the value function. Equation (2.1) was then used to find the appropriate points on the value function. Then the maximizations in equations (2.2), (2.3), and (2.4) were solved to find a new strategy set or price set, according to Nash bargaining r· '.., based on the new value function. This process continued until the strategy set (ie. individual match price levels) converged and an equilibrium was found for those parameter values. A new level of the mone, supply was then specified and the procedure repeated. For all the equilibria found, the value function, V(m), was concave and strictly monotonically increasing in m. Equilibria in which V(m) was not concave may exist but were not considered at this time. Also, the search for equilibria primarily involved pure strategies for the individual match prices. Mixed strategy equilibria did exist (especially in areas where pure strategy equilibria did not seem to occur) but were generally difficult to identify given the large number of matches and therefore possible mixed strategies. As such, there were some money supplies for which no equilibria were found. However, more than enough were found to be able to discuss money neutrality and inflation taxes.

Equilibria were found most easily for small values of x so the simulations began by assuming x=.05. First, consider a model with M=4. This model had 19 matches that had to be examined to determine the individual price levels. The remaining matches had, by default, price levels of one money unit because one of the agents either had only one unit in inventory with which to pay or could only accept one unit before the inventory maximum was reached. As a result of the trade offs that could be made between the prices for individual matches, there tended to be multiple equilibria in the larger models for some money supplies. This made it difficult to find equilibria for certain money supplies since the simulations was very sensitive to starting values. A more intensive search would undoubtedly have filled in the gaps in the equilibria but was left to a later date.

When agents were permitted to hold only zero or one unit of money, the aggregate price level was constant at one for all values of the money supply. There was only one match for which there could be any bargaining; a double coincidence of wants match between the agent with a good and an agent with a good and one unit of money. In this match the agents optimally chose a pure barter trade instead of having the second agent give up the unit of money as well. As a result, the aggregate price level for a good stayed fixed at one for all $S \in (0,1)$.

With prices determined as outlined above, a one unit maximum on money holdings yields a result far from neutrality. This basic search model with severe inventory restrictions cannot yield changes in the price level when the money supply changes. Agents are not able to freely change the prices at which they trade because their money holdings are restricted and come in indivisible quantities. Therefore, the price level is fixed when $M=1$ and the model is highly non-neutral.

Table 2.1 lists the aggregate price levels when money holdings are restricted to two, four, and eight units. Certain price levels are marked with an asterisk to signify that there were changes in the negotiated prices of individual matches from those which prevailed at the next lower money supply. These changes in the strategy set led to jumps in the general price level while, between changes in the strategy set, P varied in a smoother fashion. Figures 2.1 to 2.3 plot the price levels for $M=2$, $M=4$, and $M=8$, respectively.

When agents were allowed to hold up to two discrete units of money there was some variation in the price level. Initially all prices were set at one but as the money supply rose further there was a change in the price level for one of the matches that caused the aggregate price to start increasing. However, the price level did not rise much and eventually began to fall even as S rose further. All the variation in the aggregate price level was generated by differences in the individual price levels in double coincidence of wants matches (Single coincidence match prices were all constant at one money unit). A change in the money supply would shift the weights on the various double coincidence of wants matches and thus change the aggregate price level. As the money supply rose toward its maximum, the number of agents with two units of money approached one. As such, money trades became increasingly difficult because fewer agents could accept money. Eventually there were primarily only pure barter trades occurring and the price level fell back toward one. This was true for any value of M. As S approached its maximum of M, the price level would first increase but eventually fall again toward P=1.

When the maximum allowed money holdings were again doubled to four units the price level began to behave in a more desirable manner. Three columns of prices are shown for this example of the model in Table 2.1 to emphasize the existence of multiple equilibria and the distinct differences between them with respect to prices. The aggregate price level increased with the money supply up to S=3.30 and declined thereafter. However, certain individual matches may have experienced price level decreases as S and the general price level rose. This was especially possible when numerous individual price level changes occurred as did among the three columns shown.

The largest model simulated allowed agents to hold up to eight discrete units of money. For this example especially there were gaps in the simulated price levels shown in Table 2.1 that illustrate the difficulty with which equilibria were found. Further analysis with respect to mixed strategies would undoubtedly have identified equilibria over the full range of possible S values, but this would not have added substantially to the overall results. For M=8, the price level was increasing in the money supply for most of its range. For the highest values of the money supply, as in the smaller models, the price level became decreasing in S. This resulted because agents are not as free to select their prices when they hold close to the maximum permitted inventory of money. If everyone holds the maximum permissible money supply then money will not be used in trading and all trades will use barter on a one for one basis.

Tables 2.2a and 2.2b give the percentage change in the aggregate price level, P, when there was a doubling, or 100% change, in the nominal money supply S. The first column represents the initial money supply before the change in S so, for example, the row at S=0.50 is the percentage change in price when S increased from 0.50 to 1.0. Table 2.2a shows the percentage change in P when the inventory restrictions are held fixed while Table 2.2b allows the maximum inventory level to double with the money supply. Where two numbers are shown, there were multiple equilibria at either the initial or final S value.

Note that this analysis is comparing steady state equilibria and does not explain the adjustment between equilibria. As such, if there are multiple equilibria at the new money supply, the model cannot say which is most likely to result after the change. The equilibrium which does result will be decided by how agents chose to change their beliefs in response to the increase in money supply. This model does not attempt to explain this. If there are multiple equilibria at the higher money supply level then the model can only predict a possible range for the change in the aggregate price level. Other equilibria may in fact exist which could yield a result closer to money neutrality than shown here.

Consider the change in S from 0.25 to 0.50. With the inventory maximum fixed at one, there was no change in the price level. However, if the inventory maximum is relaxed from one to two at the same time as the doubling in S, there is a 1.080% increase in price. Similarly, beginning with a model with a two unit maximum, there were increases of 0.326% and 1.666% in P when the inventory limit was either held fixed or doubled, respectively. Starting from a model in which agents hold up to four units, there are 2.605% and 1.734% increases, respectively, in P depending on whether the inventory restrictions changed or not. In general, the experiment which allowed the maximum inventory holdings to double with the money supply yielded higher percentage changes in the price level than when the restrictions were fixed. Higher initial money supply values also tended to yield percentage changes in P that were larger. However, there were smaller changes in P at the highest initial S values as the price level became decreasing in the money supply.

Some of the highest percentage changes in P were found when the money supply was between S=1.50 and S=1.80 and agents were initially allowed to hold only up to two units of money. When the maximum allowable money inventories were doubled with the money supply, the change in the price level was just over 90% and thus quite close to money neutrality. Similarly, with S=3.2 and agents initially holding up to four units of money, the price level increased 95% when S doubled. There was even an isolated example of an increase in the price level of greater than 100%.

In general, however, all the changes in the aggregate price level were far below 100% so the models were a long way from yielding money neutrality. The percentage changes in P were larger when the inventory maximums were doubled along with S but were still often small. The changes in P were also more substantial when starting from higher initial money supplies. The smallest model, with agents only holding up to one unit of money was very far from being neutral. However, when the inventory restrictions were relaxed further there were at least some observations that were much closer to neutrality. Although still far from exhibiting neutrality, the model did show some promising results for surprisingly small relaxations of the inventory restrictions. If the model were to be enlarged even further, the results would undoubtedly appear even better.

Welfare (W), shown below, was calculated as the weighted average of the various

$$W = \sum_{m=0}^{M} \pi(m)V(m)$$

value functions using the population fractions as weights. For the model with M=1, welfare increased to its maximum at S=0.50 and then declined for larger money supplies. Welfare was also concave in nature for the larger models. For M=2 the maximum welfare was again at one-half of the inventory maximum (S=1.0). For the models assuming M=4 and M=8, however, welfare was maximized at values of S that were less than one-half of the maximum holdings. Agents' welfare increased as more money was added to the economy and raised the rate at which they could trade with one another. However, as the money supply rose further, monetary trade became more and more difficult because agents' inventories and hence their ability to trade freely was restricted due to the storage technology. This caused overall welfare to eventually decline as the money supply grew.

The model in which agents could only hold one unit of money consistently yielded the lowest utility. However, depending primarily on the initial money supply, welfare did not necessarily increase monotonically as the inventory restrictions were relaxed. For a given money supply, if the storage technology was relaxed to allow for greater inventories then agents were not necessarily made better off even though they were freer to choose the price levels at which they traded. However, if the initial money supply was small then adding equivalent increments to the inventory grid (say by adding six units in moving from the two unit maximum model to the eight unit model) did not really benefit the agents because there was too little money to take advantage of the new freedom. Starting from a model with two unit maximum inventory, if money was made more divisible so that it came in fractional quantities but had the same maximum amount then agents would likely be better off than if money still came in one unit quantities but agents could hold up to eight units. This was especially true for small levels of the money supply. The fineness of the grid for money holdings (and therefore prices) relative to the initial money supply was the important factor in determining whether agents could benefit from simply relaxing the inventory and indivisibility assumptions on money.

With this section of the chapter we have shown once again the importance to the search model of money of assuming that agents are restricted by the storage technology in terms of how much can be held in inventory. It is not surprising that, with controls on the holding of money, money is not neutral in the model. The chapter shows that it is possible to introduce a Nash bargaining mechanism for the explicit determination of prices in the search model of money.

2.5 Inflation Tax:

A tax on the use of money can easily be added to this model in the same manner as proposed by Kiyotaki and Wright (1991b). Assume there is a government which has two types of representatives, those which tax agents' money holdings and those which purchase goods from agents. With an arrival rate of τ, private agents will meet a government representative who will confiscate one unit of money from their inventory. The government pools the money it collects and has its purchasing agent redistribute it by buying goods. With an arrival rate of γ, agents will encounter a government purchasing agent who will pay one unit of money for the private individual's good. Of course, money can only be taxed from an agent with positive inventory holdings and, similarly, no good can be purchased from an agent who already holds the maximum amount of money possible. Assume the government will balance its budget so that the rate of collection of money equals its rate of redistribution. The government budget constraint is given by

$$\tau \sum_{m=1}^{M} \pi(m) = \gamma \sum_{m=0}^{M-1} \pi(m) \tag{2.17}$$

where $\pi(m)$ is the fraction of agents in the population with m units of money in inventory. In the simulations, τ was set exogenously but γ was determined endogenously to balance the government budget constraint.

The taxing of money in this manner is basically an inflation tax and, as such, the price level P should increase with the tax rate τ. In a steady state equilibrium, however, the price level is constant given a fixed level of τ greater than zero. There would be a one-time change in P if τ increased instead of any on-going change for a given τ. Prices were calculated excluding transactions involving the government.

The data in Table 2.3 for S=0.25 show that P did increase when τ was small and when there were changes in the price levels for individual matches. For low values of τ, the price level even increased when there were no changes in the negotiated prices for individual matches. However, for higher values of the tax rate we see that the aggregate price level began to decline slowly. Figures[4] 2.4 and 2.5 show that, although the price level was increasing in τ in a stepwise fashion, there were numerous cases when it actually declined with the tax rate. In fact, with x=0.05 and S=3.75 the price level was decreasing in τ throughout. When the price level for a particular match changed it was always to increase with τ. However, an increase in the tax rate also tended to shift the distribution of agents. At times, this readjustment was able to lower the price level, even

[4] The figures show price levels for the model in which agents can hold up to four units of money since it was much easier to find equilibria for this example. The other models behaved in a similar manner.

in the face of rising prices for individual matches. When an increase in τ was able to move more agents into the tails of the distribution, ie. have more agents with no money or M units of money, then the aggregate price level was positively correlated with the inflation tax rate (even when there were no changes in the price strategy set). However, if agents were shifted into the centre of the money inventory distribution, where individual match prices tended to be lower, then the price level fell as τ increased (given no changes in the strategy set).

A positive correlation between τ and P was more likely for higher values of x and lower values of S and τ. In other words, when low values of x and high values of S restricted trading opportunities it was possible that the aggregate price level was negatively related to an inflation tax. When agents were restricted by the money inventory maximum, P could fall with τ. If the inventory restrictions were removed, as in the model of Section III, then P and τ would likely be positively related for all parameter values.

The Kiyotaki and Wright model also yielded a price level that was positively related to the tax rate, τ. Their increasing price level resulted from a declining real money supply or number of agents holding one real unit of money (given a fixed nominal money supply). With government revenue calculated as τ times the fraction of agents with money, they were also able to generate a Laffer curve for government revenue. Revenue initially rose with τ and then fell as the decline in money holders began to dominate.

In this model, however, government tax revenue is the left side of equation (2.17) where the number of agents with nominal money, not real money, is important. Movements in this fraction of agents were always dominated by the increase in the tax rate τ so government revenue was always increasing in that tax rate. No Laffer curve existed in government revenues.

Overall welfare, expressed as the weighted average of the relevant value function points, was inversely related to the price level as τ was varied. Thus, in general, welfare declined as the tax rate increased. Note that, with production being costless and instantaneous, a tax on money has no real effects on this model economy because the aggregate supply of goods is always the same. If the government's aim was to collect a certain amount of goods for itself, then it could do so with no real or nominal effect by simply confiscating the required goods directly instead of purchasing them. Of course, if production was neither costless nor instantaneous, both the inflation tax and a direct commodity tax would have real and nominal effects.

2.6 Comparative Statics:

Some of the parameters were varied in order to get a flavour for the comparative static properties of the model. The parameter β represents the arrival rate of trading partners for an agent. Higher values of β, not surprisingly, yielded greater welfare levels as agents were able to trade more often. The price level, however, was non-increasing in β. If the price strategy set did not change with β, then there was no change in the general price level even though there were changes in the population fractions and thus the weights used in P. This tended to occur more for higher S values. If the strategy set did change, then the price level tended to fall as β increased. Changes in β did not affect the general price level directly because it could be completely factored out during the calculation of P. As mentioned, β also did not affect P indirectly through the population fractions when the strategy set was fixed. The arrival rate of trading partners could only affect the general price level by changing the price strategy set, which tended to move in the opposite direction. A higher value of β would improve the future opportunities of the agents and thus decrease the importance of a particular match. As such, the price level can be expected to decline as β and the rate of trading rises.

The parameter x represents the fraction of agents in the economy who produce a good that a given agent can consume. An increase in x would thus raise the supply of goods available for that agent's consumption. Conversely, it is also the fraction of individuals who will demand the agent's own production good and so any increase in x would augment this demand. With both these supply and demand side factors, the effect of a change in x upon P was ambiguous.

Given the existence of multiple steady state equilibria for many parameter values, the overall price level might increase or decrease depending on which new steady state actually emerged. In examples for which there was no change in the strategy set of individual price levels, the overall price level might have risen or fallen, or both, after an increase in x. Figure 2.6 plots the price level against x for three different money supplies. It seemed that for higher money supply values the general price level was decreasing in x while for low values of S it was increasing. For some S values, P would both rise and fall with x even without any change in the strategy price set for individual matches. This can be seen in Figure 2.6 for S=3.0 from x=0.05 to x=0.40.

The probability of a double coincidence of wants in any given match is x^2 whereas the probability of a single coincidence is 2x(1-x). Thus, when x increases, there is a greater probability of a double coincidence but a lower probability of a single coincidence if x > 1/2 (or a higher probability if x < 1/2). The number of double coincidence of wants trades increased with x but the number of single coincidence trades varied in an ambiguous fashion. There was however a definite increase in the relative number of double coincidence of wants trades. The prices for individual matches were both increasing and decreasing in x depending on the particular level of the other parameters. With increases in x shifting trading toward the relatively higher priced double coincidence of wants trades, it was possible for the general price level to rise. However, this was not always the case. Most new trades resulting from an increase in x were pure barter trades. This tended to decrease, P_{gm}, the average price level for barter trades. For low S values the effect of a greater weight on higher priced barter trades dominated and P rose with x. For high money supply values, the decline in P_{gm} dominated the increase in the number of barter trades and P declined with x.

2.7 Conclusions:

The inventory restrictions on the holding of money are relaxed in this model so that a Nash bargaining game can be introduced as the mechanism through which prices are determined. The price level that is determined depends crucially upon the assets (money and goods) possessed by the two individuals involved in a particular match. There will not be a price that prevails in all matches although an aggregate price level can be computed.

The model illustrates that in a search model of money, with agents bargaining to determine prices, money neutrality does not hold when there are inventory restrictions. As these restrictions are relaxed, the model moves towards money neutrality albeit quite slowly. If the asset restrictions are relaxed completely to allow for unbounded and divisible holdings, money does become neutral. Although this model may not exhibit the desirable property of money neutrality having a more explicit mechanism for explaining the setting of prices is an important improvement on previous work. Prices are determined within each individual match through a Nash bargaining game played by the agents involved. The price selected will depend on whether or not there is a double coincidence of wants and on the amount of wealth held in inventory by the agents. An agent could receive one price for his production good today and a completely different price for the same good tomorrow. A law of one price will not hold in this model since each match is essentially independent of behaviour in the other matches so that only the wealth and characteristics of the agents directly involved has any bearing on the negotiated price.

Whether or not the model yielded a positive relationship between price and the inflation tax rate depended on the parameter values assumed. The less restricted was trade, ie. a high number of trading partners and low money supply, the more likely was a positive relationship between price and the tax rate. The model did not yield a Laffer curve for government tax revenue.

Appendix 2.1

$$rV(m) = \beta \sum_{m'=0}^{M} \pi(m') \, [\, x^2 \,(\, u + V(\tilde{m}(m,m')) - V(m))$$
$$+ x(1-x)\,(\, u + V(\tilde{m}_1(m,m')) - V(m)) + (1-x)x\,(\, V(\tilde{m}_2(m,m')) - V(m))] \tag{2.1}$$

$$0 \le \tilde{m}_1 < m \le M, \quad 0 \le m < \tilde{m}_2 \le M, \quad 0 \le \tilde{m} \le M$$

Bargaining:

$$\tilde{m}(m,m'): \{0, 1, \ldots, M\}^2 \rightarrow \{0, 1, \ldots, M\}$$
$$\tilde{m}_1(m,m'): \{0, 1, \ldots, M\}^2 \rightarrow \{0, 1, \ldots, M\}$$
$$\tilde{m}_2(m,m'): \{0, 1, \ldots, M\}^2 \rightarrow \{0, 1, \ldots, M\}$$

where:

1. Double coincidence of wants:

$$\tilde{m}(m,m') = \underset{\tilde{m}}{\operatorname{argmax}} \, [u + V(\tilde{m}(m,m')) - V(m)][u + V(m+m'-\tilde{m}(m,m')) - V(m')] \tag{2.2}$$

where: $m, m', \tilde{m}, m+m'-\tilde{m} \in \{0,1,\ldots,M\}$

2. Single Coincidence: First agent wants other's good

$$\tilde{m}_1(m,m') = \underset{\tilde{m}_1}{\operatorname{argmax}} \, [u + V(\tilde{m}_1(m,m')) - V(m)][V(m+m'-\tilde{m}_1(m,m')) - V(m')\,] \tag{2.3}$$

where: $m, m', \tilde{m}_1, m+m'-\tilde{m}_1 \in \{0,1,\ldots,M\}$

3. Single Coincidence: Second agent wants other's good

$$\tilde{m}_2(m,m') = \underset{\tilde{m}_2}{\operatorname{argmax}} \, [V(\tilde{m}_2(m,m')) - V(m)][u + V(m+m'-\tilde{m}_2(m,m')) - V(m')\,] \tag{2.4}$$

where: $m, m', \tilde{m}_2, m+m'-\tilde{m}_2 \in \{0,1,\ldots,M\}$

The following equation governs the steady state population distribution of agents for all $a \in \{0,1,2, \ldots, M\}$.

$$\sum_{m=0}^{M} \sum_{m'=0}^{M} \beta x^2 \pi(m) \pi(m') \; [I(\tilde{m}(m,m') = a) \; + \; I(m+m'-\tilde{m}(m,m') = a) \;]$$

$$+ \sum_{m=0}^{M} \sum_{m'=0}^{M} \beta x(1-x) \pi(m) \pi(m') \; [I(\tilde{m}_1(m,m') = a) \; + \; I(m+m'-\tilde{m}_1(m,m') = a) \;]$$

$$+ \sum_{m=0}^{M} \sum_{m'=0}^{M} \beta (1-x) x \pi(m) \pi(m') \; [I(\tilde{m}_2(m,m') = a) \; + \; I(m+m'-\tilde{m}_2(m,m') = a) \;] \tag{2.5}$$

$$- \sum_{m'=0}^{M} \beta x(2-x) \pi(a) \pi(m') \quad = \quad 0$$

where: $I(z) = \begin{cases} 1 & \text{if condition z is true} \\ 0 & \text{otherwise} \end{cases}$

Appendix 2.2

Value Function:

$$rV(m) = \beta\int_0^{\infty}\{\, x^2[u + V(\tilde{m}(m,m')) - V(m)\,] + x(1-x)[u + V(\tilde{m}_1(m,m')) - V(m)\,] + (1-x)x[\,V(\tilde{m}_2(m,m')) - V(m)\,]\,\}\, d\Pi(m') \tag{2.6}$$

where:

$$\tilde{m}(m,m') = \underset{m,\ m',\ \tilde{m},\ m+m'-\tilde{m}\ \in\ [0,\infty)}{argmax}\ [u + V(\tilde{m}(m,m')) - V(m)][u + V(m+m'-\tilde{m}(m,m')) - V(m')] \tag{2.7}$$

$$\tilde{m}_1(m,m') = \underset{m,\ m',\ \tilde{m}_1,\ m+m'-\tilde{m}_1\ \in\ [0,\infty)}{argmax}\ [u + V(\tilde{m}_1(m,m')) - V(m)][\,V(m+m'-\tilde{m}_1(m,m')) - V(m')] \tag{2.8}$$

$$\tilde{m}_2(m,m') = \underset{m,\ m',\ \tilde{m}_2,\ m+m'-\tilde{m}_2\ \in\ [0,\infty)}{argmax}\ [\,V(\tilde{m}_2(m,m')) - V(m)][u + V(m+m'-\tilde{m}_2(m,m')) - V(m')] \tag{2.9}$$

Define $\pi(m)$ as the population density function and $\Pi(m)$ as the population distribution function. The following steady state equation will determine the density function for all $a \geq 0$.

$$
\begin{aligned}
&\beta x^2 \ \{ \ [1-\Pi(a)]^2 \ [\ Pr(\tilde{m}(m,m') \leq a) \ + \ Pr(m+m'-\tilde{m}(m,m') \leq a) \ ] \\
&\qquad + [1-\Pi(a)] \ \Pi(a) \ Pr(\tilde{m}(m,m') \leq a) \ \} \\
&+ \ \beta x(1-x) \ \{ \ [\ 1-\Pi(a)]^2 \ Pr(\tilde{m}_1(m,m') \leq a) \ + \ [\ 1-\Pi(a) \ ] \ \Pi(a) \ Pr(\tilde{m}_1(m,m') \leq a) \ \} \\
&+ \ \beta(1-x)x \ \{ \ [\ 1-\Pi(a) \ ]^2 \ Pr(m+m'-\tilde{m}_2(m,m') \leq a) \\
&\qquad\qquad + \Pi(a) \ [\ 1-\Pi(a) \ ] \ Pr(m+m'-\tilde{m}_2(m,m') \leq a) \ \} \\
&- \ \beta x^2 \ \{ \ [\ 1-\Pi(a) \ ] \ \Pi(a) \ Pr(m+m'-\tilde{m}(m,m') > a) \\
&\qquad\qquad + \Pi(a)^2 \ [\ Pr(\tilde{m}(m,m') > a) \ + \ Pr(m+m'-\tilde{m}(m,m') > a) \ ] \ \} \\
&- \ \beta x(1-x) \ \{ \ [\ 1-\Pi(a) \ ] \ \Pi(a) \ Pr(m+m'-\tilde{m}_1(m,m') > a) \\
&\qquad\qquad + \Pi(a)^2 \ Pr(m+m'-\tilde{m}_1(m,m') > a) \ \} \\
&- \ \beta(1-x)x \ \{ \ \Pi(a)^2 \ Pr(\tilde{m}_2(m,m') > a) \ + \ \Pi(a) \ [\ 1-\Pi(a) \ ] \ Pr(\tilde{m}_2(m,m') > a) \ \} \ = \ 0
\end{aligned}
\qquad (2.10)
$$

Proof of Proposition 1:

1. The variable $\tilde{\mathbf{m}}$**(m,m')** solves the first order condition from equation (2.7) given below:

$$\frac{\partial Z}{\partial \tilde{m}} = V'(\tilde{m})(u + V(m+m'-\tilde{m}(m,m')) - V(m')) - V'(m+m'-\tilde{m})(u + V(\tilde{m}(m,m')) - V(m)) \leq 0$$

$$\tilde{m} \geq 0,\ m+m'-\tilde{m} \geq 0,\ \text{and}\ \ \tilde{m}\frac{\partial Z}{\partial \tilde{m}} = 0$$

Where Z represents the product of the surpluses to be maximized. The inequality in the FOC can of course be replaced by an equality if the solution $\tilde{\mathbf{m}}$**(m,m')** is interior to the feasible set. The FOC can be rearranged to form:

$$\frac{u + V(\ m+m'-\tilde{m}(m,m')\ ) - V(m')}{u + V(\ \tilde{m}(m,m')\ ) - V(m)} \leq \frac{V'(\ m+m'-\tilde{m}(m,m')\ )}{V'(\ \tilde{m}(m,m')\ )} \tag{2.11}$$

But through the substitution of V(m)=Y(2m) and 2$\tilde{\mathbf{m}}$**(m,m')**=$\tilde{\mathbf{m}}$**(2m,2m')** it can be shown that $\tilde{\mathbf{m}}$**(2m,2m')** satisfies

$$\frac{u + Y(\ 2(m+m')-\bar{m}(2m,2m')\ ) - Y(2m')}{u + Y(\ \bar{m}(2m,2m')\ ) - Y(2m)} \leq \frac{Y'(\ 2(m+m')-\bar{m}(2m,2m')\ )}{Y'(\ \bar{m}(2m,2m')\ )} \tag{2.12}$$

which is simply the FOC for the problem

$$\bar{m}(2m,2m') = \mathbf{argmax}\,[u + Y(\bar{m}(2m,2m')) - Y(2m)][u + Y(2(m+m')-\bar{m}(2m,2m')) - Y(2m')] \tag{2.13}$$

Therefore $\bar{m}(2m,2m')$ satisfies the equilibrium condition for the Nash bargaining problem solved by the maximizing agents with inventories of 2m and 2m'. Similarly, it can be shown that $\bar{m}_1(2m,2m')$ and $\bar{m}_2(2m,2m')$ solve the Nash bargaining problems for single coincidence of wants trades at the new money supply.

2. By the definition of the value functions we have that

$$\begin{aligned} rY(2m) \;=\; rV(m) \;=\; \beta\int_0^{\infty} \pi(m')\,\{\, & x^2[u + V(\bar{m}(m,m')) - V(m)\,] \\ & + x(1-x)[u + V(\bar{m}_1(m,m')) - V(m)\,] \\ & + (1-x)x[\,V(\bar{m}_2(m,m')) - V(m)\,]\,\}\,dm' \end{aligned} \tag{2.14}$$

Through the substitution in the right-hand side of Y(•), $\overline{m}$(•), and $2\theta(2a)=\pi(a)$ for all $a\geq 0$ we can derive the equation

$$\begin{aligned} rY(2m) \;=\; \beta\int_0^{\infty} \theta(2m')\,\{\, & x^2[u + Y(\bar{m}(2m,2m')) - Y(2m)\,] \\ & + x(1-x)[u + Y(\bar{m}_1(2m,2m')) - Y(2m)\,] \\ & + (1-x)x[\,Y(\bar{m}_2(2m,2m')) - Y(2m)\,]\,\}\,dm' \end{aligned} \tag{2.15}$$

But this is simply the definition of the equilibrium value function similar to equation (2.6) but based on the new price levels and distribution function. Therefore, the new value function defined by Y(2m)=V(m) satisfies the equilibrium condition.

3. Finally, the population density function, $\theta(m)$, and distribution function, $\Theta(m)$, can also be shown to satisfy the relevant equilibrium condition. Beginning with equation (2.10) and making use of the new prices and that, by construction, $2\theta(2a)=\pi(a)$ for all $a\geq 0$, we can derive the following equation

$$
\begin{aligned}
&\beta x^2 \{ [1-\Theta(2a)]^2 [Pr(\bar{m}(2m,2m')\leq 2a) + Pr(m+m'-\bar{m}(2m,2m')\leq 2a)] \\
&\qquad + [1-\Theta(2a)]\, \Theta(2a)\, Pr(\bar{m}(2m,2m')\leq 2a) \} \\
&+ \beta x(1-x) \{[1-\Theta(2a)]^2 Pr(\bar{m}_1(2m,2m')\leq 2a) + [1-\Theta(2a)]\Theta(2a) Pr(\bar{m}_1(2m,2m')\leq 2a)\} \\
&+ \beta(1-x)x \{ [1-\Theta(2a)]^2 Pr(m+m'-\bar{m}_2(2m,2m')\leq 2a) \\
&\qquad + \Theta(2a) [1-\Theta(2a)] Pr(m+m'-\bar{m}_2(2m,2m')\leq 2a) \} \\
&- \beta x^2 \{ [1-\Theta(2a)]\, \Theta(2a)\, Pr(m+m'-\bar{m}(2m,2m')>2a) \\
&\qquad + \Theta(2a)^2 [Pr(\bar{m}(2m,2m')>2a) + Pr(m+m'-\bar{m}(2m,2m')>2a)] \} \\
&- \beta x(1-x) \{ [1-\Theta(2a)]\, \Theta(2a)\, Pr(m+m'-\bar{m}_1(2m,2m')>2a) \\
&\qquad + \Theta(2a)^2 Pr(m+m'-\bar{m}_1(2m,2m')>2a) \} \\
&- \beta(1-x)x \{\Theta(2a)^2 Pr(\bar{m}_2(2m,2m')>2a) + \Theta(2a))[1-\Theta(a)] Pr(\bar{m}_2(2m,2m')>2a) \} = 0
\end{aligned}
\tag{2.16}
$$

But this would define a steady state for the distribution of agents at the new money supply by equating the rightward flow of agents with the leftward flow of agents at every non-negative money supply.

Therefore, it has been shown that defining $Y(m)$, $\theta(m)$, $\bar{\mathbf{m}}(\mathbf{m},\mathbf{m}')$, $\bar{\mathbf{m}}_1(\mathbf{m},\mathbf{m}')$, and $\bar{\mathbf{m}}_2(\mathbf{m},\mathbf{m}')$ in a manner that exhibits neutrality will yield equations that satisfy the appropriate equilibrium conditions and form an equilibrium at the new, higher, money supply. The model thus exhibits money neutrality when there is no inventory restriction on the holding of money.

Table 2.1:
Equilibrium Aggregate Price: P

Money Supply (S)	2 Unit Max.	4 Unit Max.			8 Unit Max.
0.05	1.0	1.00011			1.00020
0.10	1.0	1.00096*			1.00097
0.15	1.0	1.00235*			1.00245*
0.20	1.0	1.00434			**
0.25	1.00752*	1.00685			1.00761*
0.30	1.00838	1.00978			1.01128*
0.35	1.00912	1.01306			1.01567*
0.40	1.00976	1.01662			1.02077
0.45	1.01031	1.02038			1.02657
0.50	1.01080	1.02431			1.03308
0.55	1.01121	1.02835			1.04028
0.60	1.01157	1.04068*			1.04817
0.65	1.01188	1.04482			1.05692*
0.70	1.01214	1.04895			**
0.75	1.01235	1.05306			1.08374*
0.80	1.01252	1.05713	1.14274*		1.09423
0.85	1.01266	1.06115	1.15214		**
0.90	1.01275	1.06510	1.16146		**
0.95	1.01280	1.06899	1.17070		1.22537
1.00	1.01282	1.07279	1.17986		1.24161
1.10	1.01275	1.08181*	1.19796		**
1.20	1.01252	1.08890	1.21577		**
1.30	1.01235	-	1.23328*		1.24694*
1.40	1.01157	-	1.25052		1.29285*
1.50	1.01080	-	1.26749		1.33034
1.60	1.00976	-	1.28420		**
1.70	1.00838	-	1.30067		1.57213*
1.80	1.00652	-	1.31689		1.64031*
1.90	1.00393	-	1.33287	1.55647*	1.68627
2.00	-	-	1.34862	1.58161	1.73210
2.10	-	-	-	1.60284	1.77785*
2.20	-	-	-	1.62414	**

* Strategy set changes from previous S level.

** No equilibrium was found although at least one, probably a mixed strategy, undoubtedly existed.

Table 2.1 Continued:
Equilibrium Aggregate Price: P

Money Supply (S)	2 Unit Max.	4 Unit Max.			8 Unit Max.
2.30	-	-	-	1.64551	**
2.40	-	-	-	1.66694	**
2.50	-	-	-	1.68840	**
2.60	-	-	-	1.70987	**
2.70	-	-	-	1.73128	**
2.80	-	-	-	1.90886*	**
2.90	-	-	-	1.91907	**
3.00	-	-	-	1.92769	**
3.10	-	-	-	1.93448	**
3.20	-	-	-	1.93910	**
3.30	-	-	-	1.94101	**
3.40	-	-	-	1.93938	**
3.50	-	-	-	1.93281	**
3.60	-	-	-	1.91871	**
3.70	-	-	-	1.89169	2.73427*
3.80	-	-	-	1.83818	2.76937
3.90	-	-	-	**	2.80366
4.00	-	-	-	-	**
6.40	-	-	-	-	3.78282
6.50	-	-	-	-	3.78202
6.60	-	-	-	-	3.77987
7.00	-	-	-	-	3.74940
7.10	-	-	-	-	3.73521
7.20	-	-	-	-	**
7.30	-	-	-	-	3.69333
7.40	-	-	-	-	3.66322
7.50	-	-	-	-	3.45482
7.60	-	-	-	-	3.35570
7.70	-	-	-	-	3.20400
7.80	-	-	-	-	2.95143

* Strategy set changes from previous S level.

** No equilibrium was found although at least one, probably a mixed strategy, undoubtedly existed.

Table 2.2a: Percentage Change in P when Nominal Money (S) Increases 100%: No Change in Inventory Maximum

Initial Nominal Money (S)	1 Unit max.	2 Unit max.	4 Unit max.	8 Unit max.
0.05	0	0.0	0.085	0.077
0.10	0	0.0	0.338	**
0.15	0	0.838	0.741	0.881
0.20	0	0.976	1.223	**
0.25	0	0.326	1.734	2.528
0.30	0	0.316	3.060	3.648
0.35	0	0.299	3.543	**
0.40	0	0.273	3.985/12.406	7.197
0.45	0	0.242	4.383/13.826	**
0.50	0	0.200	4.733/15.186	20.185
0.55	-	0.152	5.199/16.493	**
0.60	-	0.094	4.634/16.826	**
0.65	-	0.027	18.038	17.979
0.70	-	-0.006	19.216	**
0.75	-	-0.153	20.387	22.755
0.80	-	-0.273	21.480/12.379	**
0.85	-	-0.423	22.572/12.892	**
0.90	-	-0.615	23.640/13.832	**
0.95	-	-0.876	24.685/13.852	37.613
1.00	-	-	25.511/14.303	39.504
1.10	-	-	50.132/35.575	**
1.20	-	-	53.085/37.110	**
1.30	-	-	38.644	**
1.40	-	-	52.645	**
1.50	-	-	52.087	**
1.60	-	-	50.997	**
1.70	-	-	49.106	**
1.80	-	-	45.700	**
1.90	-	-	37.911	**
3.70	-	-	-	33.974
3.80	-	-	-	21.183
3.90	-	-	-	5.271

** Not available due to missing equilibria.

Table 2.2b: Percentage Change in P when Nominal Money (S) Increases 100%: Inventory Maximum also Doubles

Initial Nominal Money (S)	1 to 2 Units	2 to 4 Units	4 to 8 Units
0.05	0.0	0.096	0.086
0.10	0.0	0.434	**
0.15	0.838	0.978	0.891
0.20	0.976	1.662	1.636
0.25	1.080	1.666	2.605
0.30	1.157	3.203	3.802
0.35	1.214	3.947	**
0.40	1.252	4.691/13.695	7.634
0.45	1.275	5.424/14.962	**
0.50	1.282	6.133/16.725	21.214
0.55	-	6.982/18.468	**
0.60	-	7.645/20.186	**
0.65	-	21.880	19.345
0.70	-	23.552	23.252
0.75	-	25.203	26.331
0.80	-	26.832	**
0.85	-	28.441	48.153/36.453
0.90	-	30.031	54.005/41.228
0.95	-	31.602	57.744/44.039
1.00	-	33.155	61.458/49.531
1.10	-	60.369	**
1.20	-	64.633	**
1.30	-	68.935	**
1.40	-	88.703	**
1.50	-	90.709	**
1.60	-	92.036	**
1.70	-	92.326	**
1.80	-	90.628	**
1.90	-	83.098	107.775
3.20	-	-	95.081
3.30	-	-	94.737
3.40	-	-	**
3.50	-	-	93.987
3.60	-	-	**
3.70	-	-	93.648
3.80	-	-	82.555
3.90	-	-	**

** Not available due to missing equilibria.

Table 2.3:
Aggregate Price under an Inflation Tax: S=0.25

τ	2 Units	4 Units	8 Units
0.0	1.00752	1.00685	1.00761
0.05	1.00757	1.00705	1.00794
0.10	1.00760	1.00721	1.00822
0.15	1.00762	1.00732	1.00845
0.20	1.00764	1.03273	1.03424
0.25	1.00766	1.03302	1.03455
0.30	1.00767	1.04653	1.04838
0.35	**	1.04655	1.05040
0.40	1.20236	**	**
0.45	1.19788	**	1.27523
0.50	1.19419	1.26446	1.27053
0.55	1.19120	1.26066	1.26657
0.60	1.18846	1.25739	1.26313
0.65	1.18620	1.25454	1.26024
0.70	1.18423	1.25203	1.25808
0.75	1.18249	1.24982	1.25773
0.80	1.18096	**	**
0.85	1.17959	**	**
0.90	1.17836	**	**
0.95	1.17726	1.29570	1.30706
1.00	1.17625	1.29337	1.30460

* Strategy set changes from previous τ level.

** No equilibrium was found although at least one, probably a mixed strategy, undoubtedly existed.

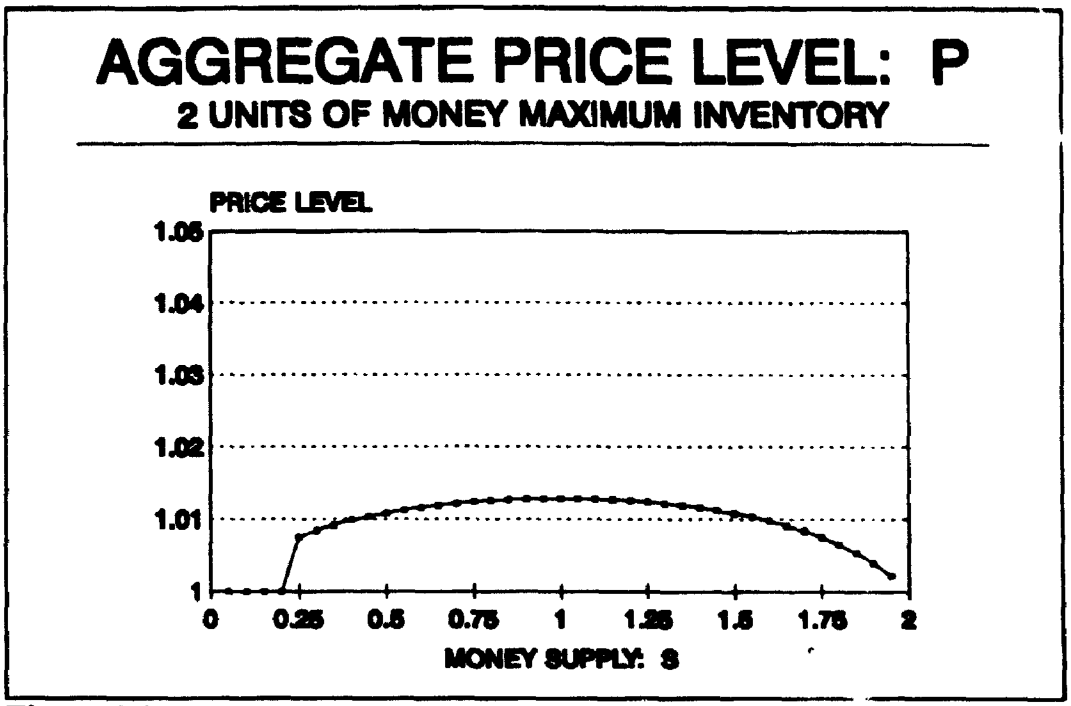

Figure 2.1

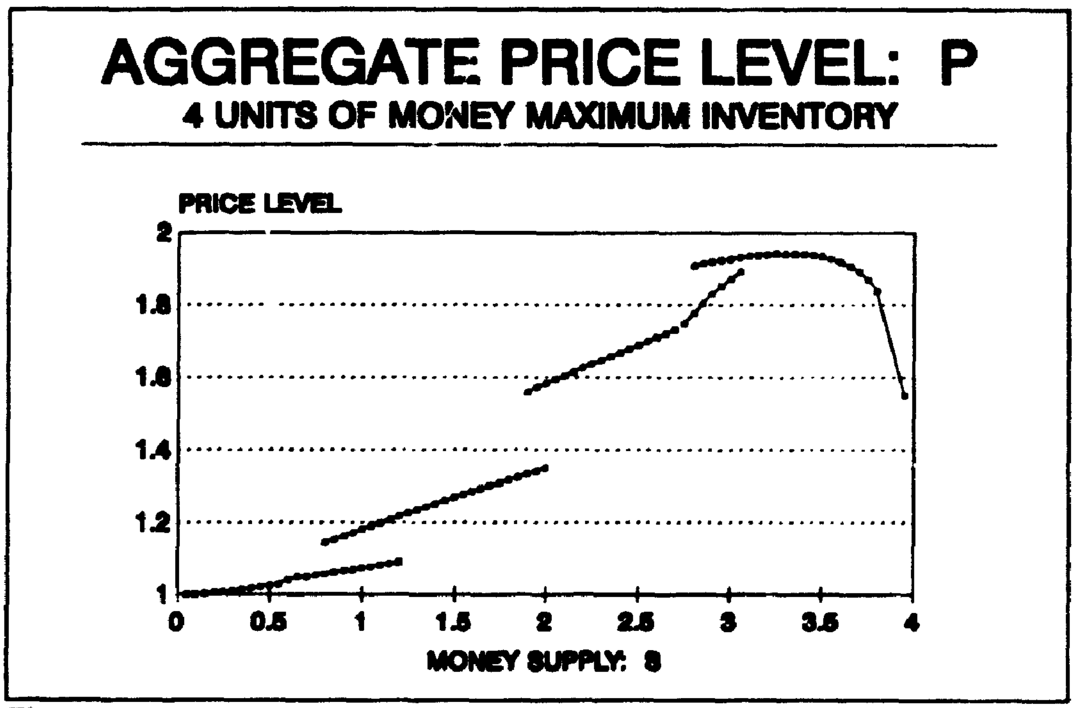

Figure 2.2

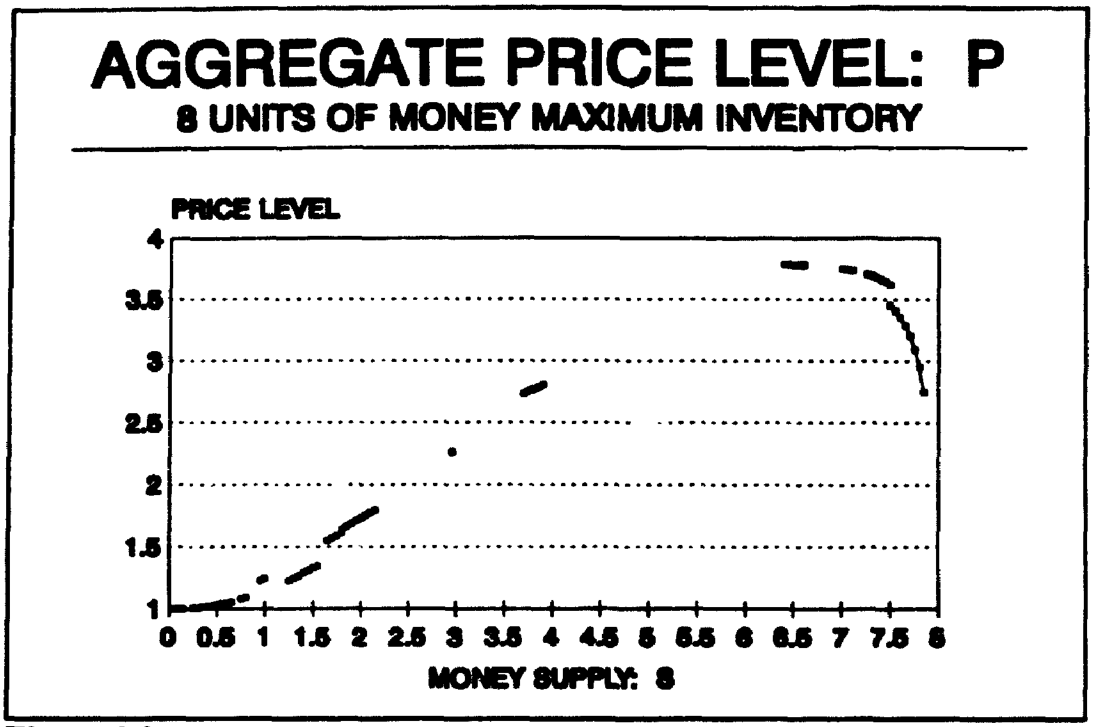
AGGREGATE PRICE LEVEL: P
8 UNITS OF MONEY MAXIMUM INVENTORY
PRICE LEVEL
4
3.5
3
2.5
2
1.5
1
0 0.5 1 1.5 2 2.5 3 3.5 4 4.5 5 5.5 6 6.5 7 7.5 8
MONEY SUPPLY: S

Figure 2.3

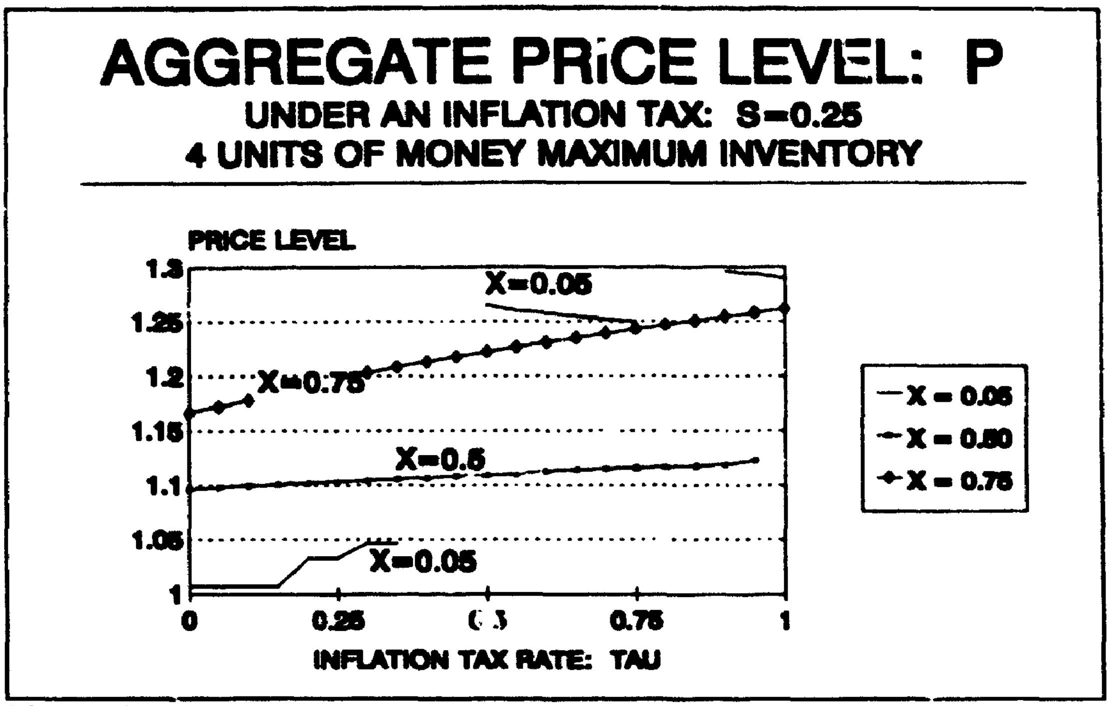

Figure 2.4

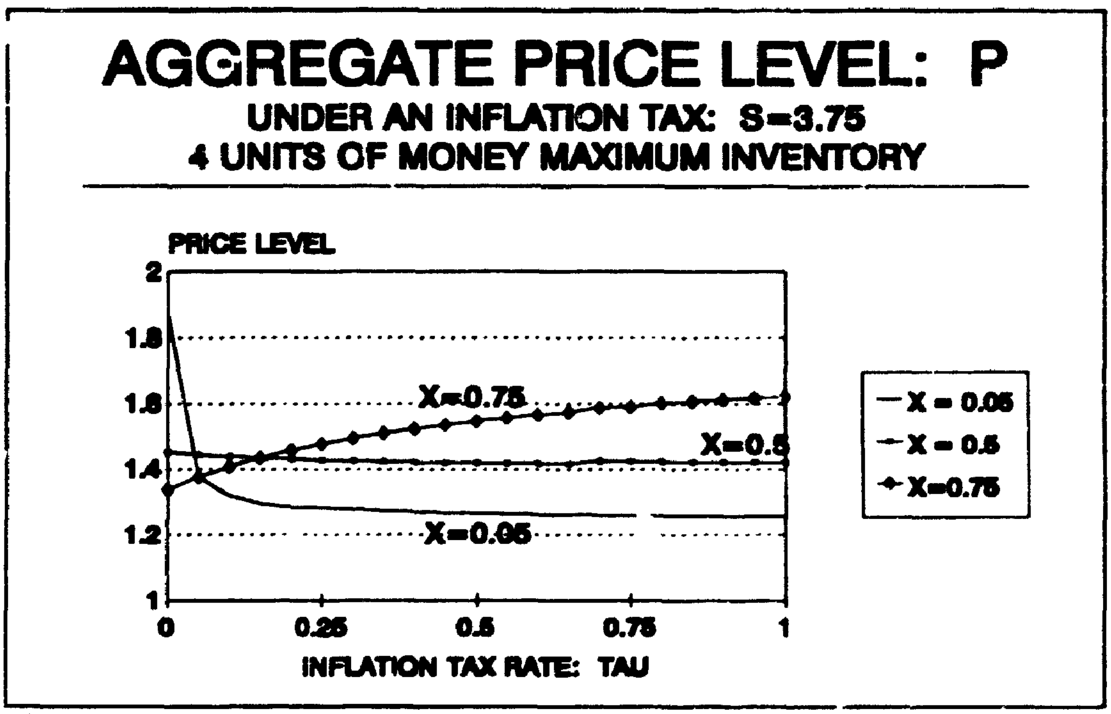

Figure 2.5

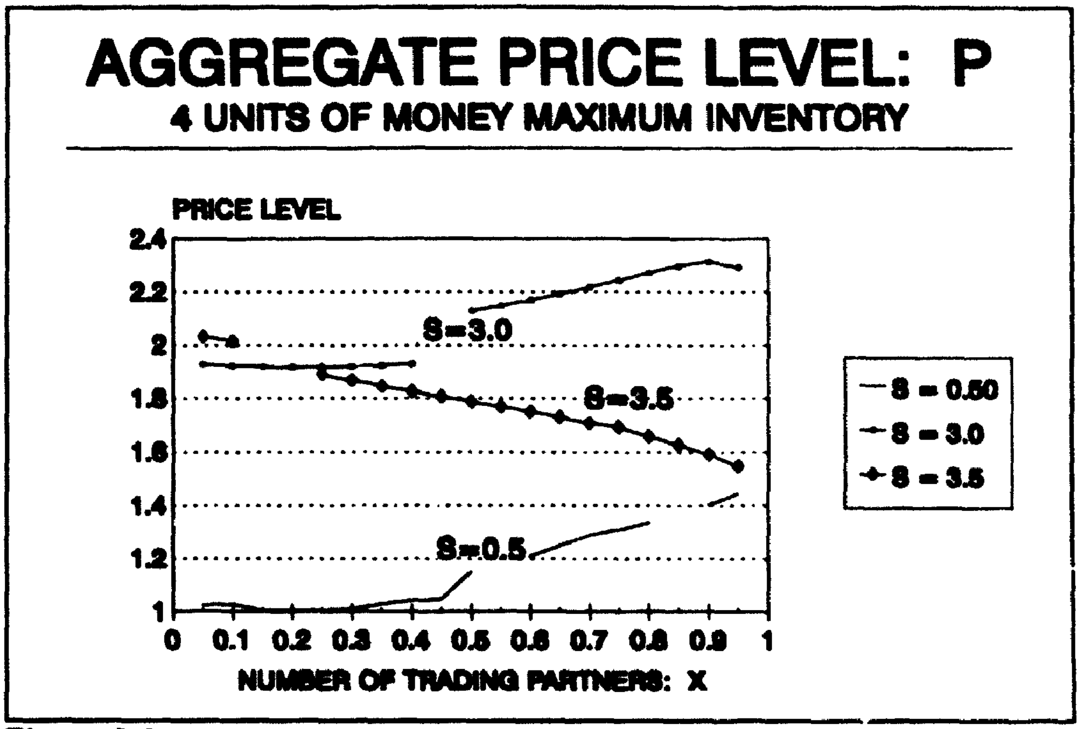
AGGREGATE PRICE LEVEL: P
4 UNITS OF MONEY MAXIMUM INVENTORY
PRICE LEVEL
2.4
2.2
2
1.8
1.6
1.4
1.2
1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
NUMBER OF TRADING PARTNERS: X
S=3.0
S=3.5
S=0.5
S = 0.50
S = 3.0
S = 3.5

Figure 2.6

CHAPTER 3: ENDOGENOUS MONEY AND GOODS PRODUCTION IN A SEARCH MODEL

3.1 Introduction:

Many monetary models define money as an intrinsically useless, unbacked, and exogenously given fiat currency. At many times in history, however, people have used commodities as media of exchange. The intrinsic properties of these commodities have long been discussed (see Jevons (1877)) but little effort has been made to incorporate them into the basic structure of monetary models. Kiyotaki and Wright (1989) have done some work on modelling the emergence of a commodity money as a medium of exchange in a search model. It was generally true in that model that the commodity chosen as the medium of exchange was that commodity with the lowest storage cost. There were, however, also examples of a 'speculative equilibrium' in which certain agents sometimes chose a high storage cost good as the medium of exchange in addition to the low cost good.

The purpose of this chapter is to further expand this area of the literature that studies commodity money. A potential medium of exchange will be endowed with certain intrinsic properties, other than just the storage costs analyzed by Kiyotaki and

Wright (1989), and only under specific conditions will it emerge as an endogenous and generally accepted money. Monetary production decisions are endogenous in this model so that agents can choose not only which commodity to use as money but also how much they wish to have circulate in the economy.

Jevons (1877) has provided probably one of the best and earliest descriptions of the properties that a money must possess if it is to circulate as a medium of exchange. These characteristics can be summarized briefly as follows: utility, portability, indestructibility, homogeneity, divisibility, stability of value, and cognizability. These properties are fairly evident. Jevons argued that the commodities that emerge as media of exchange must also provide some direct utility. Gold, for instance, provides utility when used in ornamentation. He believed that habit or convention could maintain an established money in equilibrium but that initially agents would need to value the commodity for reasons other than its exchange services. Commodity monies need to be easily carried or transferred between agents. They must be relatively indestructible or durable and be of homogeneous quality. The commodities must be divisible so that they can be used for transactions of all sizes. There must be a stable value so that agents can be certain that it won't be dropped from the economy while held in inventory. Finally, the agents must be able to recognize the commodity and its value if they are to accept it as a medium of exchange.

After Jevons (1877) and Menger (1892), little work seems to have been done on the emergence of money as a medium of exchange and its inherent characteristics until Brunner and Meltzer (1971), Jones (1976), King and Plosser (1986), and Kiyotaki and

Wright (1989). Some of the great economists of the early 20th century, notably Keynes (1936), Hicks (1967), Samuelson (1947), and Wicksell (1935), have discussed the general properties of money and the history of its evolution into its modern form. These authors reiterated the same basic characteristics of money proposed by Jevons and others before him back to Adam Smith in 1776. This work however tended to be from a broad perspective and involved little in the way of formal modelling.

These authors and many[5] since then have completed much work studying the relative merits of commodity money systems (the Gold Standard) and fiat money economies. The general purpose of this work has been to determine whether commodity monies really can yield a more stable price system or to explain why commodity monies are more expensive in terms of resources and hence are less efficient than fiat monies. The Sargent and Wallace (1983) paper is probably one of the more comprehensive works in terms of formal modelling of commodity money and consideration of the main policy questions. Although their model does not entail an endogenous medium of exchange (it uses the OLG framework), it does determine the endogenous supply of gold. They show that the commodity money equilibrium is inefficient because of the resources used to produce gold. There can also be two commodity monies in their model if they possess equal depreciation rates. The commodity monies with higher depreciation rates will have a zero supply in equilibrium. This chapter generates a similar result since any commodity with higher production, storage, and depreciation costs will never circulate as an

[5] See Friedman (1960,1963), Sargent and Wallace (1986), Nickelsburg (1985), and Fischer (1986) to list a few.

endogenous medium of exchange. If there are multiple commodity monies, then each will have lower relative costs of one type but higher costs of another.

Sargent and Wallace (1983) model the depreciation of gold as the only intrinsic property of the commodity money while Nickelsburg (1985) emphasizes the durability of the commodity backing a government issued representative money. Fischer (1986) and Whitaker (1979) model commodity monies without any consideration of their characteristics. They focus on policy issues that come into importance once the market has already selected the media of exchange in some manner. Only with the more recent literature that focuses on explaining the emergence of money in an endogenous framework has serious consideration been given once again to the properties of money. Brunner and Meltzer (1971) emphasize the low relative marginal cost of acquiring information about money in comparison to other commodities while Jones (1976) focuses on the relative degree of acceptability of a commodity in determining whether it will emerge as a medium of exchange. King and Plosser (1986) hypothesize that if the quality of gold is verifiable in a world of generally uncertain qualities then it may emerge as a medium of exchange. Finally, Kiyotaki and Wright (1989), upon which my work is most closely based, study the storage costs of commodities and the beliefs of the agents as the important factors determining whether a particular commodity could circulate as a money.

One of the purposes of this chapter is to further expand the list of characteristics of money that have been studied in a framework of endogenous media of exchange. The production cost of money, although alluded to as an important factor by Keynes (1936) and Wicksell (1935), does not seem to have previously been formally modelled. This cost is an important factor in this chapter for determining the supply of money and the value of accepting money in trade. The depreciation rates and storage costs of money have been considered by various authors as discussed above but never together within a single model.

Although a number of the models discussed above generate an endogenous supply of money (see Fischer (1986), Sargent and Wallace (1983), King and Plosser (1986), and Kiyotaki and Wright (1989)), they do not mention the possibility of an overproduction of money that is a basic result of my model. In fact, King and Plosser (1986) assert that the competitive equilibrium is equivalent to the social planners optimal allocation in their framework. As long ago as Adam Smith (1776) some economists believed that there could be an excess supply of private money if agents were free to produce it. Smith observed that the banks of his era tended to over-issue their private bank notes which were backed by gold. He argued, however, that this was not a sustainable policy and the banks would either fail or be forced to reduce their issues of bank notes. Therefore, there could be no equilibrium with an oversupply of money even though agents may attempt to achieve this. In contrast, it is possible, using the search model of this chapter, to generate a sustainable steady state equilibrium in which agents produce too much money and, as a result, lower their overall welfare.

Until the recent work explaining the endogenous emergence of money as a medium of exchange, much of the work involving commodity monies has focused on the issue of whether such a money is less efficient and just how many resources are used up by such a system. They have generally taken the existence of the commodity money for granted and have only discussed in a broad and general sense the evolution or emergence of the money. This thesis here is a part of the recent but expanding literature beginning at a more fundamental level and explaining first why there may be a commodity money before examining the questions regarding its relative merits.

This chapter does not attempt to examine all the possible characteristics that a money should exhibit but instead concentrates on only a few properties. The commodities considered for money in this model will be of homogeneous quality, have a stable value since all prices are fixed at one, and will be easily recognized by all agents. For simplicity, assume money is indivisible. The properties most closely modeled in this chapter are portability, through the introduction of storage costs, and indestructibility, by subjecting all commodities to exogenous depreciation. The production of all goods and commodity monies is assumed to require the payment of fixed costs. These can be used to approximate the scarcity of a commodity. Scarcer products have higher production costs and quite often provide higher value.

Some of the basic predictions of the model are given here. Only potential monies that are more costly per unit to produce than consumption goods will circulate as media of exchange. This is interpreted as implying that commodity monies must be sufficiently scarce in order to be valued. Even some commodities that have greater costs in all categories than consumption goods can be used as money provided that agents are willing to do so and believe that others will also accept them in trade. Two media of exchange cannot simultaneously circulate if onc strictly dominates the other in terms of lower costs of production, storage, and depreciation. If the costs of one commodity money decline, this can force agents to stop using certain types of other commodity monies. This can be thought of as being opposite to a traditional Gresham's Law since in this model the 'good' money (ie. the money with lower costs) can drive out the bad monies. A generally accepted commodity money can coexist with a fiat money even though the fiat money may have strictly lower costs. The advantage of being able to produce the commodity money yourself induces agents to use the more costly commodity money rather than wait to find the fiat money.

If the double coincidence of wants problem is sufficiently large then the production of the commodity money will be welfare improving for the agents. The money supply and output may be positively or negatively correlated in response to changes in the parameters of the model. The correlation was positive only if the initial level of goods inventories was sufficiently high.

The next section of the chapter outlines the basic model with only one type of commodity money and no fiat money. Section 3.3 examines government intervention to attempt to maximize welfare. Section 3.4 models an environment with multiple media of exchange while Section 3.5 examines the interaction of a commodity money and a fiat money. The final section provides a summary and conclusion to the chapter.

3.2 The Model:

The model will follow those used previously in the literature, primarily that of Kiyotaki and Wright (1990). Their model can be adapted quite readily to consider the issues described above. Assume there is a unit mass of infinitely lived agents uniformly distributed on a circle of circumference two. Let there also exist a continuum of goods that are indexed by the points of the same circle. An agent's type is identified by a particular good on the circle. Agents are assumed to receive utility from consuming any good within an exogenously specified distance $x \in [0,1]$ of the good that defines their type. For simplicity, let agents receive a fixed utility of u from consuming any good within this distance x of their type.

Assume also that there is a transaction cost of $\varepsilon > 0$ in utility terms that must be paid by any agent who receives a consumption good in trade. This is useful in ruling out the possibility of indirect barter trades between agents. Since it is commodity monies that are primarily being considered here, a money trade is essentially an indirect barter trade except that the intermediate step always uses the same commodity. The ε rules out indirect barter trades other than those using the proposed commodity money. This is one advantage that commodities used as media of exchange have over consumption goods. Without a transaction cost, the production cost structure of the model would ensure that no consumption good emerged as a generally accepted medium of exchange. Without $\varepsilon > 0$, however, degenerate equilibria with no goods production or mixed strategy equilibria with consumption good media of exchange are possible. These equilibria are not interesting given the goal of this chapter so the transaction cost is used to rule them out. This issue will be discussed in more detail below.

Assume now that an agent's production good is known and is chosen from an independent uniform distribution on the same circle as the consumption goods. Therefore, there is a probability x that a randomly selected agent with a good will have something that a particular agent will consume. Similarly, there is a probability x that a particular agent's production good is consumable by another randomly selected agent. This implies a meeting between two agents, both of whom have goods, will result in a double coincidence of wants with probability x^2.

Agents are assumed not to be able or willing to consume the good that they produce themselves. As a result, they must search for trading partners in order to acquire something they will be able to consume. Agents are assumed to meet pairwise and randomly according to a Poisson process with a fixed arrival rate of $\beta>0$. Assume that there is no mechanism by which credit may arise in the model.

Agents are assumed to be able to store only one unit of one commodity at any given time (ie. agents with a unit of a good cannot also store another good or money). A storage cost must be paid to hold a commodity in inventory. The flow disutilities, s_m and s_g, are paid to store money and consumption goods, respectively. These can be thought of as storage costs net of any flow utility agents may experience from storing the commodity. If $s_m>0$, then agents pay a cost to store money. However, if $s_m<0$, then agents receive a flow utility from simply holding the commodity money. A negative storage cost on money could approximate either a rate of return or possibly, as proposed by Jevons, the direct utility required by a commodity in order for it to serve as a medium of exchange. There will be monetary equilibria regardless of whether s_m is greater or smaller than s_g provided that the difference is not too large.

Assume that each agent is capable of producing both his own production good and another commodity called gold. Every agent has access to the same production technology for gold. Goods are commodities that yield utility when consumed by certain agents in the economy while gold is assumed to yield no consumption utility but may circulate as a medium of exchange if the agents choose. This model shows that a commodity may serve as a medium of exchange even if it does not yield direct utility. This is contrary to what was proposed by Jevons. The level of consumption utility provided by a commodity may affect whether or not the agents would initially consider it as a commodity money. Utility, however, was not a requirement in this framework once agents began to use it as a medium of exchange. The benefit of improved trading opportunities outweighed any need for utility from using the money.

An agent cannot produce both goods and gold simultaneously due to the inventory restrictions that exist. Hence, if there are to be both money and goods produced in equilibrium, the agents must be indifferent between producing the two commodities. Let the variable θ represent the probability with which agents choose to produce goods and $1-\theta$ the probability they produce money. The return from each activity will be identical in an equilibrium with money.

Production is assumed to be instantaneous after the payment of a fixed cost which is measured in terms of utility. All consumption goods are subject to the same fixed production cost, $c_g>0$. A cost of $c_m>0$ must be forgone by an agent in order to produce a unit of gold. Since production is instantaneous an agent will always be able to participate in the trading market.

We shall consider equilibria in which there is an on going production of both goods and money. Since gold is not consumed by any agent its supply would thus tend to increase over time given agents are continuously producing it. Thus, to ensure that the money supply simply does not increase to its maximum value, assume that gold is subject to an exogenous depreciation rate to approximate the rate at which it wears out through its use as a medium of exchange. Let the Poisson arrival rate $\gamma_m>0$ denote the rate at which this occurs. Gold may be used fully as a medium of exchange up to that point that it perishes. After that, the gold becomes completely unusable and the agent holding it at that time is returned to production.[6]

Let the arrival rate $\gamma_g\geq0$ represent the depreciation rate for consumption goods. A good yields full utility until the time that it depreciates. After that, it is valueless to everyone and will be thrown away. The agent must then decide once again whether to produce money or goods. It is quite interesting to note that there are equilibria for this model in which the agents may endogenously choose to use a commodity as a medium of exchange that is less durable than the goods the money is used to purchase. The gold's usefulness as a medium of exchange can outweigh the cost of depreciation.

One of the innovations of this chapter is that agents are now able to produce the money used as a medium of exchange. Since agents are free to choose the amount of their production, the supply of gold is endogenously determined within the model.

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[6] There may also be equilibria in which all of the gold desired by the agents is produced in the first instant of time. After that agents would produce only goods. For this type of equilibrium to be a steady state there could be no depreciation of gold because this would slowly erode its supply over time until none remained. This type of equilibrium will not be considered here, since we wish to examine steady state equilibria and the characteristics of money (including depreciation).

Agents chose not only whether or not there will be a medium of exchange but also the supply of that money which will circulate in the system. Agents can only hold zero or one unit of money, so the aggregate money supply will simply be the fraction of agents in the system who have gold in inventory. Let m represent this fraction. The only other state in which an agent may be found is a goods trading state (given by the fraction g=1-m) when the agent has one unit of his production good in inventory.

The next step in defining the model is to set up the value functions for the agents in each state. Let r denote the private rate of time preference. The value of having a single unit of a consumption good in inventory is then given by V_g in equation (3.1). This agent type can perform double coincidence of wants barter trades for another good or it can trade for money. With an arrival rate of β, the agent meets trading partners.

$$rV_g = \beta g x^2 [u - \epsilon + \max_{\theta} \{\theta(V_g - c_g) + (1-\theta)(V_m - c_m)\} - V_g] + \beta m x \max_{\pi} \pi (V_m - V_g)$$
$$+ \gamma_g [\max_{\theta} \{\theta(V_g - c_g) + (1-\theta)(V_m - c_m)\} - V_g] - s_g + \dot{V}_g \tag{3.1}$$

With probability g, the partner also has a good in inventory and, with probability x^2, there is a double coincidence of wants so that the agents will trade. The agent consumes the new good for a net utility of u-ε and then must decide once again whether to produce a unit of gold or another unit of the production good. A fraction θ of the agents (or similarly a fraction θ of the times an agent must produce over his lifetime) will produce a new good for a return of $(V_g - c_g)$ while the remaining agents will produce a unit of money for a return of $(V_m - c_m)$. The expected capital loss from producing after having been a goods trader is thus $[\theta(V_g - c_g) + (1-\theta)(V_m - c_m) - V_g]$. This shows plainly why an equilibrium with both money and goods requires identical returns to production or $(V_g - c_g) = (V_m - c_m)$. If $(V_g - c_g) > (V_m - c_m)$ then only goods are produced and if the opposite is true then only money is produced.

Agents will not participate in indirect barter trades for consumption goods because all goods provide the same trading opportunities and the transaction cost makes for a negative return to accepting a non-consumable good.

There is a probability mx that the trading partner has money, or gold, instead of a good and also wants the agent's good. Then the agent must decide with probability π whether or not to accept the money in trade for his good. If the gold is accepted, then the agent moves into the money holding state with a net capital gain of $V_m - V_g$. In a monetary equilibrium with a universally accepted medium of exchange π will equal 1 signifying that money is always accepted in trade. This model cannot generate mixed strategy monetary equilibria with $0<\pi<1$ unless production costs for money and goods are identical, ie. $c_m = c_g$.

The rate at which consumer goods perish or depreciate is approximated by the arrival rate γ_g. After a good depreciates the agent must either produce a new good or a unit of money in order to continue trading. Depreciation represents a capital loss to the agents since the fixed production costs must to be paid again. The variable s_g represents the flow storage cost paid by the agents in order to hold their production good in inventory.

Let $\dot{V}_g$ represent the rate of change of V_g over time. This will be zero in the steady state equilibria to be examined in this chapter. It will be argued later that the model's equilibrium will be stable and tend toward this steady state.

The value of holding money is represented by V_m in equation (3.2) while $\dot{V}_m$ represents the rate of change of this value. Agents with money will also meet trading partners at the rate β. With probability $gx\Pi$ (the upper case Π represents the strategy of

$$\begin{aligned} rV_m &= \beta gx\Pi\left[\, u-\epsilon + \max_{\theta}\{\theta(V_g-c_g)+(1-\theta)(V_m-c_m)\} - V_m\right] \\ &\quad + \gamma_m\left[\max_{\theta}\{\theta(V_g-c_g)+(1-\theta)(V_m-c_m)\} - V_m\right] - s_m + \dot{V}_m \end{aligned} \tag{3.2}$$

the other agents) the arriving agent will be a goods trader storing a good that the money holder wants and who is willing to trade for gold. After the trade, the money holder immediately consumes the new good for a utility of u-ε net of transaction costs and then produces either a good or another unit of money. With probability θ the agent will produce a unit of his production good while, with probability 1-θ, a unit of gold is produced. The gold will depreciate at rate γ_m after which the money holder suffers a capital loss and must once again produce something.

To close the model, it is necessary to specify the laws of motion governing the population fractions of agents. Using the identity 1=g+m only one law of motion need be specified. There is a net outflow of agents from the goods trading state by those goods traders who produce money after barter trades or goods depreciation. The outflow of goods traders who accept money is also partially offset by those money traders who produce goods after giving up their money. Therefore, after barter trades (which occur at rate βg^2x^2), the depreciation of goods (occurring at rate $g\gamma_g$), and money trades (occurring at rate $\beta gmx\pi$) there is a (1-θ) probability of an outflow of an agent from the goods trading state. The inflow of agents to the goods trading state is by money traders who produce a good after their money has depreciated. This occurs at rate $m\gamma_m\theta$.

$$\dot{g} = -[\beta g^2x^2 + \beta gmx\pi + g\gamma_g](1-\theta) + m\gamma_m\theta \tag{3.3}$$

Only the steady state equilibria of this model will be considered. This implies that, in an equilibrium, the rate of change of the population fractions is zero so that g is calculated by setting equation (3.3) to zero.

A pure strategy steady state monetary equilibrium will specify a production strategy of $0<\theta<1$, a money acceptance probability of $\pi=1$, a distribution of agents by inventories (g and m), and a pair of value functions, V_g and V_m, such that: i) the return to accepting money (V_m-V_g) is positive and the agents are indifferent between money and goods production, ie. $(V_m-c_m)=(V_g-c_g)$, given g, m, V_g and V_m, and ii) given π and θ, the population fractions (g and m) are in steady state (ie. equation (3.3) is zero) and V_g and V_m represent the optimized values of storing goods and money, respectively. A barter equilibrium will be characterized by a production strategy of $\theta=1$ (ie. a dominance of goods production over money production $(V_g-c_g)>(V_m-c_m)$) and a refusal to accept money (ie. $\pi=0$) because (V_m-V_g) is negative.

Assume that the initial inventories of money and goods are zero. It is possible that the economy can jump immediately to the steady state equilibrium since production decisions are made before any trade occurs and because we are considering rational expectations equilibria.

Equation (3.4) represents the return, $(V_m\text{-}V_g)$, for a goods trader who sells his inventory for money. It must be assumed that utility, net of transaction costs and expected production costs, is positive, ie. $[u\text{-}\varepsilon\text{-}\theta c_g\text{-}(1\text{-}\theta)c_m]>0$. If this restriction were violated then there would never be any production or trading and a degenerate equilibrium would result. There will at least always be barter trades when the restriction is true. The return to accepting money in equation (3.4) will then be positive when agents believe in the value of money (π=1) and money depreciation and storage costs are smaller or at least not too much larger than the comparable goods costs.

$$[r+\theta(\beta gx\pi+\gamma_m)+(1-\theta)(\beta gx^2+\gamma_g)+\beta mx\pi](V_m-V_g) \tag{3.4}$$
$$=\beta gx(\pi-x)[u-\epsilon-\theta c_g-(1-\theta)c_m]-(\gamma_m-\gamma_g)[\theta c_g+(1-\theta)c_m]+s_g-s_m+\dot{V}_m-\dot{V}_g$$

However, equation (3.4) is not the only expression that defines the equilibrium return to accepting money. Recall that in order for there to be an equilibrium with both money and goods production agents must be indifferent between producing the two types of commodities. This implies $(V_m\text{-}c_m)=(V_g\text{-}c_g)$ or $(V_m\text{-}V_g)=(c_m\text{-}c_g)$. The equilibrium return to accepting money is defined by the difference in the fixed production costs of money and goods. There will be a positive return when $c_m>c_g$ or when there is a saving of production costs by an agent producing a good and then trading for money instead of directly producing money. When this occurs the goods traders will set π=1 and always trade for the commodity money when the opportunity arises. Only commodities that are more costly to produce, or possibly scarcer, than consumption goods can serve as generally accepted media of exchange.[7]

The agent's production decision will be determined by the difference between the

$$[r+\theta(\beta gx\pi+\gamma_m)+(1-\theta)(\beta gx^2+\gamma_g)+\beta mx\pi][(V_m-c_m)-(V_g-c_g)] \tag{3.5}$$
$$=\beta gx(\pi-x)[u-\epsilon-c_g]-\gamma_m c_m-\gamma_g c_g+s_g-s_m-(r+\beta x\pi)(c_m-c_g)+\dot{V}_m-\dot{V}_g$$

gain from producing the commodity money and the gain from producing goods, $(V_m-c_m)-(V_g-c_g)$, represented in equation (3.5). If it is positive then agents produce only money ($\theta=0$) while if it is negative they produce only goods ($\theta=1$). Only if agents are indifferent will they produce both money and goods ($0<\theta<1$).

Substituting $(V_m-V_g)=(c_m-c_g)$ into equation (3.4) or (3.5), then setting $\dot{V}_m-\dot{V}_g=0$ and

$$g^* = \frac{s_m-s_g+(r+\beta x\pi)(c_m-c_g)+\gamma_m c_m-\gamma_g c_g}{\beta x(\pi-x)(u-\epsilon-c_g)}$$
$$m^* = \frac{\beta x(\pi-x)(u-\epsilon-c_g)+s_g-s_m-(r+\beta x\pi)(c_m-c_g)+\gamma_g c_g-\gamma_m c_m}{\beta x(\pi-x)(u-\epsilon-c_g)} \tag{3.6}$$

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[7] Consider once again the importance of $\varepsilon>0$. This parameter is used to rule out consumption goods as media of exchange. Suppose that all agents suddenly have access to the technology to produce a particular consumption good because it will not be considered as a potential medium of exchange (otherwise there is only a measure zero of agents who can produce the money). Because $V_m-V_g=c_m-c_g$ in equilibrium, a consumption good medium of exchange would yield $V_m-V_g=0$ since $c_m=c_g$ for this good. There can thus be no consumption good (or any other commodity) with $c_m=c_g$ circulating as a generally accepted medium of exchange. However, if $\varepsilon=0$, $c_m=c_g$, $s_m=s_g$, and $\gamma_m=\gamma_g$ then equation (4) yields a zero return to accepting money if either i) $g=0$ for any π or ii) $\pi=x$ for any g. If $g=0$ there is a degenerate equilibrium with no trade and every agent storing money. If $\pi=x$, then for any g and θ, the consumption good will circulate as a partially acceptable medium of exchange in equilibrium. Any $\varepsilon>0$ will rule out these indeterminant equilibria and allow us to consider equilibria with generally accepted media of exchange.

solving for the steady state equilibrium fraction of agents with goods and money, yields the expressions in equation (3.6). The sum of the two equilibrium fractions is of course one. The money supply or fraction of agents with money in the economy responds negatively to increases in money storage or production costs or money's depreciation rate and positively to increases in the storage costs or depreciation rate on goods. Changes in the other parameters yield ambiguous results. For low values of x, m^* is increasing in x but for high values of x it declines as x rises. The velocity of barter trades is $\beta g x^2$ while the velocity of money trades is $\beta g x \pi$. For low x the number of money trades increases relatively faster as x rises. Consequently, agents favour having more money to facilitate trade so m^* is increasing in x. When x is high however, the number of barter trades increases faster with x so agents favour barter and equilibrium m^* decreases.

Equation (3.3) can now be used to define the production strategy θ that will

$$\theta^* = \frac{\beta g^2 x^2 + \beta g m x \pi + g \gamma_g}{\beta g^2 x^2 + \beta g m x \pi + g \gamma_g + m \gamma_m} \in [0, 1] \tag{3.7}$$

achieve the equilibrium steady state mix of goods and money inventories in equation (3.6). This is given as θ^* in equation (3.7).

A steady state monetary equilibrium will exist if the agents believe that others will accept money in trade, if the cost of money production is larger than the cost of goods production, and if the population fractions in equation (3.6) are within the unit interval. If any of these conditions does not hold then a barter equilibrium will result.

Consider now what occurs when the economy first begins to operate. Suppose there is an initial level of goods inventories, say g_0. Since g^* sets equation (3.5) to zero, if $g_0>g^*$, we see that V_m-c_m>V_g-c_g initially. This prompts agents to set $\theta=0$ and produce only money. As time passes, goods are consumed or depreciate and are not replaced causing the equilibrium fraction of agents with goods to decline. Goods inventories will continue to decline until $g_t=g^*$ at some time t. At that point agents will observe that V_m-c_m=V_g-c_g and will choose $0<\theta^*<1$ to maintain g^*. The economy will continue to operate in this steady state forever if there is no change in the parameters. If $g_0<g^*$ initially, then V_m-c_m<V_g-c_g and goods inventories will rise until the steady state level of g^* is achieved.

The commodity called gold must have higher per unit production costs than the consumption goods.[8] Gold may also have a higher depreciation rate and higher storage costs than consumption goods and still be used as a medium of exchange. The service that money provides as a faster method of trading than barter can outweigh the extra costs associated with its production and use. However, if any of the costs becomes too high then the equilibrium level of the money supply in equation (3.6) is driven down to zero. Given the return to accepting money is fixed by the production costs in equilibrium, a commodity money can be driven out of the economy (say through increased storage costs) by forcing its equilibrium supply to zero even though it is generally accepted by agents (π=1) and its return is positive.

The boundaries for each parameter can easily be calculated as functions of the

[8] If agents were able to produce unrestricted and divisible quantities of goods then money would be valued only if the total production cost of money was greater than the total production cost of goods. Agents would produce until V_m-c_m = $V_g(g')$-$c_g g'$ where g' is the levels of goods produced by each agent in equilibrium.

$$c_m \in \begin{cases} (c_g, \bar{c}_m) & \text{if } s_g - s_m + (\gamma_g - \gamma_m)c_g \le 0 \\ (\underline{c}_m, \bar{c}_m) & \text{if } s_g - s_m + (\gamma_g - \gamma_m)c_g > 0 \end{cases}$$

$$\text{where: } \bar{c}_m = \frac{\beta x(\pi - x)(u - \epsilon) + (r + \beta x^2 + \gamma_g)c_g + s_g - s_m}{r + \beta x \pi + \gamma_m} \tag{3.8}$$

$$\underline{c}_m = \frac{s_g - s_m + (r + \beta x \pi + \gamma_g)c_g}{r + \beta x \pi + \gamma_m}$$

other parameters using population fraction equations in (3.6). The expression in (3.8) gives the possible range for the production cost of money. As c_m approaches $\bar{c}_m$ from below m^* is forced to zero. As c_m approaches c_g from above, the equilibrium return to accepting money goes to zero and the monetary equilibrium ceases to exist. If $c_g < \underline{c}_m$ and c_m declines too far then g^* goes to zero and there is again no monetary equilibrium. If either the depreciation rate on money or the storage cost for money increases, then both the upper and lower bounds on c_m will decline. In other words, a medium of exchange must balance high costs of one type with low costs of another type if it is to circulate in a monetary equilibrium. The response of $\bar{c}_m$ to changes in the parameters β or x was ambiguous. However, for low values of x the upper boundary on c_m tended to increase with x and decline with β (the opposite effects were true for high x values). Since increases in x make money relatively more attractive than barter for low x, $\bar{c}_m$ increases to signal that types of money which were previously inferior and unacceptable could then circulate as media of exchange. For high x values the rate of money trades is still faster than barter trades but the relative gap is smaller. As such, better quality monies are needed (ie. $\bar{c}_m$ falls) as x rises close to its maximum.

High storage costs or depreciation rates for money can also cause agents to stop using the medium of exchange in equilibrium. The upper boundaries on s_m and γ_m fall as x rises so that better quality commodities are required for media of exchange when there are more trading partners. Increases in the arrival rate of trading partners, β, have an ambiguous affect on the range of s_m and γ_m. More costly types of money (higher s_m and γ_m) become acceptable as β rises when x is low. But when there is a greater number of potential trading partners, monies with high storage and depreciation costs become unacceptable following increases in β. The basic conclusion is that the more difficult is trade using barter the wider is the range of acceptable media of exchange.

An agent's overall welfare (W) is defined as the weighted average of the value an agent receives in each state where the weights are the fractions of time spent in the states. Substituting the value functions into this expression yields the result in equation (3.9).

$$\begin{aligned} rW &= (\beta g^2x^2 + \beta gmx\pi)(u-\epsilon) - gs_g - ms_m \\ &\quad - [\beta g^2x^2 + \beta gmx\pi + g\gamma_g + m\gamma_m](\theta c_g + (1-\theta)c_m) \end{aligned} \tag{3.9}$$

The welfare maximizing level of money can then be found by differentiating with respect to m. Solving the resulting equation gives the optimal values for g and m in

$$\begin{aligned} g^{**} &= \frac{\beta x\pi(u-\epsilon-c_g) + \gamma_m c_m - \gamma_g c_g + s_m - s_g}{2\beta x(\pi-x)(u-\epsilon-c_g)} \\ m^{**} &= \frac{\beta x(\pi-2x)(u-\epsilon-c_g) - \gamma_m c_m + \gamma_g c_g - s_m + s_g}{2\beta x(\pi-x)(u-\epsilon-c_g)} \end{aligned} \tag{3.10}$$

equation (3.10). The maximizing level of the money supply will be positive if and only if the numerator is positive which is not guaranteed. If m^{**} in equation (3.10) is negative (ie. the numerator is less than zero), then redefine the optimal money supply to be zero. The higher is x the greater is the likelihood that the welfare maximizing level of m is zero. For low x, the benefit of faster trade provided by money outweighs the loss from greater production costs and permits $m^{**}>0$. However, the opposite is true for high x when the added benefit of faster trade than barter is small but the production costs are still high. There is also nothing to ensure that the economy would operate at or near this optimal level of the money supply. The actual value of m given by equation (3.6) could in fact be very different from its optimal supply. If money costs are smaller (except for production costs) or not too much larger than goods costs, then there will be an overproduction of money by the agents so that $m^{*}>m^{**}$.

In the preceding paragraph we showed that under certain circumstances the optimal supply of commodity money is non-zero. However, when do the agents themselves choose an equilibrium supply of commodity money that yields higher welfare than a pure barter equilibrium? It is straight forward to show that a commodity money equilibrium will provide higher welfare than a pure barter equilibrium if and only if $-\beta x^2(u-\varepsilon-c_g) + (r+\beta x\pi)(c_m-c_g) > 0$. In other words, commodity money can be welfare enhancing if x and $(u-\varepsilon-c_g)$ are small enough compared to (c_m-c_g). The easiest interpretation is that the double coincidence of wants problem must be sufficiently difficult (ie. x small) and the equilibrium money supply cannot be too large (ie. $u-\varepsilon-c_g$ too small and c_m-c_g too large) in order for money to be welfare enhancing.

See Table 3.1 for a summary of the comparative statics for the equilibrium and optimal population fractions. The optimal supply of money declines as the production, depreciation, and storage costs of money become larger. When x is low, m^{**} increases with x because the velocity of money rises relatively faster than the velocity of barter. This makes money more attractive. However, for high x values, the optimal money supply falls with x so that it reaches zero before x reaches one. When there is an ample supply of trading partners (high x) then there is not the need for money and agents would actually be better off by producing goods instead.

Table 3.1: Comparative Statics

	g^*	m^*	g^{**}	m^{**}
c_g	?	?	?	?
c_m	+	-	+	-
s_g	-	+	-	+
s_m	+	-	+	-
γ_g	-	+	-	+
γ_m	+	-	+	-
x	?	?	?	?

The goods output of this economy can be defined by the rate at which goods are produced. For a monetary equilibrium this would be defined by $Y_m=\beta g^2x^2 + \beta gmx\pi + \gamma_g g$ while for a barter equilibrium it would be $Y_b= \beta x^2 + \gamma_g$. The output of a monetary economy will be larger, $Y_m>Y_b$, if and only if $g^*>(\beta x^2+\gamma_g)/(\beta x(\pi-x))$. Whether this is true depends on the parameters of the model. In particular, $Y_m>Y_b$ only if $u-\varepsilon-c_g$ is sufficiently small. The addition of money to the economy may or may not increase the rate of goods production. Similarly, it can be shown that welfare in a monetary equilibrium can be larger than that of a barter equilibrium if x is small enough (barter is sufficiently difficult), $u-\varepsilon-c_g$ is small enough, or if c_m-c_g is large enough (high return to trading for money). Only if the double coincidence of wants problem is sufficiently difficult will money production be welfare increasing.

Consider now the response of output to a change in the parameters of a monetary equilibrium. Output, Y_m, may increase or decrease after a change in the goods inventories of the agents. The higher are goods inventories, g, the more likely is $\partial Y_m/\partial g<0$. If this is true then from the results of Table 3.1 we can see that the economy's output will decline following increases in the costs associated with the medium of exchange. In these circumstances a negative shock to money will decrease output. If, however, $\partial Y_m/\partial g>0$, then negative shocks to money will increase output. Therefore, the money supply and output may be either positively or negatively correlated. The two variables tend to be positively correlated if the initial level of goods inventories was sufficiently high.

From this model we can see the importance of the properties of a commodity in determining whether or not agents will believe in its value as a medium of exchange. Production costs of money must be sufficiently large for agents to value it. This can be interpreted as requiring a commodity that is sufficiently scarce to serve as a medium of exchange. Money must have low enough depreciation rates and storage costs as well. It is quite possible however to have a commodity serving as a medium of exchange that has higher costs of production, storage, and depreciation than other objects in the economy. The benefit of being a generally acceptable medium of exchange can outweigh the greater costs.

3.3 Government Intervention:

If there were a government that could influence certain parameters of the model then they could potentially act to improve the overall welfare of the agents. Assume that the government can impose a subsidy or tax or some other form of legal restriction on the production of gold so as to change the fixed cost of money production. Since production costs are in terms of utility, a tax could be thought of as, say, the introduction of a licensing procedure or legal restriction on the production of money that annoyed producers and increased their disutility of production. Both the optimal and equilibrium levels of m decline as c_m increases although the equilibrium level falls faster. Suppose that $m^*_0>m^{**}_0>0$ were the original equilibrium and optimal money supplies, respectively. The government can raise c_m to lower the equilibrium money supply to m^{**}_0. However, the higher production costs will also lower the optimal money supply to some $m^{**}_1<m^{**}_0$. Since the equilibrium money supply falls faster than the optimal value there will, however, be a point at which $m^*=m^{**}<m^{**}_0$. Let $\bar{c}_m$ represent the production cost for this point at which the equilibrium and optimum coincide. However, it is not guaranteed that this point can be reached with a positive money supply. The government may find that in chasing after the optimal money supply they only manage to drive money out of the economy even though there originally was a positive optimal money supply. This was more likely to occur the greater was the number of trading partners (x) in the economy

since the added benefit of money is smaller the more trading partners there are.

When the government first decides to change c_m in order to improve the welfare of the agents it will appear that it is possible to do so. However, changing c_m alters the equilibrium and the environment such that there is no guarantee that the new equilibrium will even provide a greater welfare level than the initial equilibrium. This is an application of the Lucas critique because the government action to improve welfare will change the economic environment and may actually decrease overall welfare. This was more likely to occur for small x values. A rational government with foresight, however, would take into account the effect it has on the economy. Such a government would select the optimal level for c_m to maximize welfare and should not fall into the trap of chasing after further illusory potential increases. The optimal production cost is given by c_m^*.

$$c_m^* = \frac{(s_g - s_m)(r+\beta x\pi) + \beta x(u-\epsilon-c_g)[\pi(r+\beta x\pi) + x\gamma_m] + (r+\beta x\pi)c_g[2(r+\beta x\pi)+\gamma_g-\gamma_m]}{2(r+\beta x\pi)(r+\beta x\pi+\gamma_m)} \tag{3.11}$$

The effect of government intervention is illustrated in Figure 3.1, assuming specific values for the parameters. This graph plots welfare as a function of the fraction of goods traders in the economy. Points A and C mark the equilibrium outcomes for money production costs of c_{m1} and c_m^*, respectively. The curve ACE joins points of equilibrium welfare for increasing levels of c_m. Points B and D are the maximum welfare levels for production cost levels of c_{m1} and c_m^*, respectively. The curve BDE joins points representing the maximum welfare for higher and higher production costs. Note how agents are overproducing money, or underproduction goods, so that there is potential for welfare to be improved by increasing goods production (assuming $c_m<\tilde{c}_m$ initially). At E, with production costs of $\tilde{c}_m$, the equilibrium outcome occurs where welfare is maximized. A naive government starting at point A will see welfare level B as its target. However, in chasing after this maximum welfare they will end up at point E which, although higher than the initial welfare, is much less than the original target. An economy starting at point C that tries to get to the optimum at point D will end up lowering its welfare level since outcome E is below the initial equilibrium.

A forward looking government starting at point A would realize the effects it was having on the economic environment and would stop raising c_m at c_m^* with an equilibrium at C instead of trying to achieve the seemingly possible further welfare increase to point D. If the initial c_m is below c_m^*, then the optimal government intervention is a tax to raise gold production costs so that the inventories of goods increase while the supply of money declines. However, if the initial c_m is above c_m^* then the optimal government policy is a subsidy to lower costs and promote the production of money.

There is a role in this model for a rational government. They can impose an optimal tax or subsidy so that money production costs are c_m^* and the overall welfare of agents is maximized. If there are a large enough number of trading partners for each agent then the optimal policy may be to use taxes to drive money out of the economy.

3.4 Multiple Types of Money:

The issue of multiple media of exchange has also been considered by Kiyotaki and Wright (1989, 1990) in a one country model and by Matsuyama, Kiyotaki, and Matsui (1992) in a two country international model. In their first paper, Kiyotaki and Wright showed that a random matching model could generate equilibria with multiple commodity monies or equilibria with a commodity money existing alongside fiat money. Aiyigari and Wallace (1991) also considered this in a more general framework with an arbitrary number of goods and mixed strategies. Subsequently, Kiyotaki and Wright (1990) provided an example in which two types of fiat currency circulated as media of exchange but with differing rates of acceptance. Even a fiat money that was dominated in terms of its flow yield of utility could circulate with a greater probability of acceptance if the agents believed that that would occur.

Matsuyama, Kiyotaki, and Matsui (1992), considers the emergence of an international fiat currency in a two country model. They derived conditions under which there would be an equilibrium with a single international currency from one of the countries, or a unified world currency, or autarky in which agents did not accept the foreign currency in trade.

This chapter's model can also be set up to include two types of money, say gold and silver, for which the supplies are endogenously, instead of exogenously, determined. Agents may endogenously choose to produce goods and each of the two types of money. For both monies to have value the agents would at least have to believe that everyone would use them both as media of exchange. Further parameter restrictions are also required and will be discussed below.

The value functions for this model can then be specified by the following equations.

$$
\begin{aligned}
rV_g \;=\; & \beta g x^2 [\, u-\epsilon + \max_{\theta} \{\, \theta_g(V_g-c_g) + \theta_{m1}(V_{m1}-c_{m1}) + \theta_{m2}(V_{m2}-c_{m2}) \,\} - V_g \,] \\
& + \beta m_1 x \max_{\pi_{m1}} \pi_{m1}(\, V_{m1} - V_g) + \beta m_2 x \max_{\pi_{m2}} \pi_{m2}(\, V_{m2} - V_g) \qquad (3.12) \\
& + \gamma_g [\, \max_{\theta} \{\, \theta_g(V_g-c_g) + \theta_{m1}(V_{m1}-c_{m1}) + \theta_{m2}(V_{m2}-c_{m2}) \,\} - V_g \,] - s_g
\end{aligned}
$$

$$
\begin{aligned}
rV_{m1} \;=\; & \beta g x \Pi_{m1} [\, u-\epsilon + \max_{\theta} \{\, \theta_g(V_g-c_g) + \theta_{m1}(V_{m1}-c_{m1}) + \theta_{m2}(V_{m2}-c_{m2})\} - V_{m1}] \\
& + \gamma_{m1} [\, \max_{\theta} \{\, \theta_g(V_g-c_g) + \theta_{m1}(V_{m1}-c_{m1}) + \theta_{m2}(V_{m2}-c_{m2}) \,\} - V_{m1} \,] - s_{m1} \qquad (3.13)
\end{aligned}
$$

$$
\begin{aligned}
rV_{m2} \;=\; & \beta g x \Pi_{m2} [\, u-\epsilon + \max_{\theta} \{\, \theta_g(V_g-c_g) + \theta_{m1}(V_{m1}-c_{m1}) + \theta_{m2}(V_{m2}-c_{m2})\} - V_{m2}] \\
& + \gamma_{m2} [\, \max_{\theta} \{\, \theta_g(V_g-c_g) + \theta_{m1}(V_{m1}-c_{m1}) + \theta_{m2}(V_{m2}-c_{m2}) \,\} - V_{m2} \,] - s_{m2} \qquad (3.14)
\end{aligned}
$$

The expected return to instantly producing something after consuming is given by $\theta_g(V_g-c_g)+\theta_{m1}(V_{m1}-c_{m1})+\theta_{m2}(V_{m2}-c_{m2})$ where θ_i is the probability with which agents produce commodity i and c_i is the appropriate fixed cost of production. Goods traders now have the option of trading for either type of medium of exchange depending on who is met through the trading process. Money traders will either trade their money for goods if they meet the proper agent or they will suffer a capital loss when the money depreciates and they are returned to production. There will be no trades of m_1 for m_2 since the agents will all have the same preference ordering between the two media of exchange.

This model can generate an equilibrium in which both types of money will circulate as media of exchange. All agents must believe that other agents will accept both monies in trade. It is also necessary that the production, storage, and depreciation costs of the two monies not be too dissimilar.

For the economic agents to produce goods and two types of money, they must be indifferent among the three possibilities. In particular, an equilibrium will be characterized by the condition $(V_g-c_g)=(V_{m1}-c_{m1})=(V_{m2}-c_{m2})$. Similarly, this fixes the equilibrium returns to accepting money in trade at $(V_{m1}-V_g)=(c_{m1}-c_g)$ for the first money type and $(V_{m2}-V_g)=(c_{m2}-c_g)$ for the second. The return to accepting money in equilibrium is once again the savings in production costs when trading for money instead of producing it yourself. An equilibrium with valued currency will hence require money production costs to be greater than goods production costs.

The equilibrium of the model is once again found by first determining the levels of goods traders and money traders that will ensure the agents are indifferent between the production options. Next, the production strategy, ie. the θ_i's, is determined which will ensure that the appropriate population fractions actually occur and are steady state values.

The equilibrium fractions of goods traders (g) and agents with type one money (m_1) can be calculated using the value functions and the equilibrium conditions. These are represented by equations (3.15) and (3.16), respectively. The identity $1=g+m_1+m_2$ is used to determine m_2.

$$g = \frac{s_{m2} - s_{m1} + (r + \gamma_{m1})(c_{m2} - c_{m1}) + (\gamma_{m2} - \gamma_{m1})c_{m2}}{\beta x(c_{m1} - c_{m2})} \tag{3.15}$$

$$\begin{aligned} m_1 = {} & (\beta x(c_{m2} - c_{m1})^2)^{-1} \\ & \cdot \{ (1-x)(u-\epsilon-c_g)[s_{m2} - s_{m1} + (r+\gamma_{m1})(c_{m2} - c_{m1}) + (\gamma_{m2} - \gamma_{m1})c_{m2}] \\ & + (c_{m2} - c_{m1})[s_{m2} - s_g + (r + \beta x)(c_{m2} - c_g) + \gamma_{m2}c_{m2} - \gamma_g c_g] \} \end{aligned} \tag{3.16}$$

Besides requiring that money production costs be greater than goods production costs, an equilibrium with two types of money will exist only if parameters are such that g and m_1 are within the unit interval.

The first important result to note is that there will be no equilibrium with two media of exchange if one type of money has strictly superior characteristics. For instance, suppose that type two money is a newer medium of exchange and that all of its costs are lower, ie. $c_{m2}<c_{m1}$, $s_{m2}<s_{m1}$, and $\gamma_{m2}<\gamma_{m1}$. From equation (3.15), this would immediately imply that the equilibrium fraction of agents with goods would be negative. Since this is not possible there could be no equilibrium with one medium of exchange being strictly dominated by another. Therefore, when agents use media of exchange they are balancing high costs of one type with low costs of another type so that overall they are indifferent between the two monies. For the remainder of the analysis assume that type one money (m_1) has higher production costs ($c_{m2}<c_{m1}$) but lower storage and depreciation costs ($s_{m2}>s_{m1}$ and $\gamma_{m2}>\gamma_{m1}$). When this is true then population fractions of agents within the unit interval is possible provided the parameters also have the proper magnitude.

Equations (3.15) and (3.16) give the levels of g and m_1 which will exist in an equilibrium. Now specify the production strategies needed to ensure that these levels will occur and will represent a steady state. The laws of motion governing the fractions of agents in each state are given in equations (3.17) and (3.18). In a steady state, the rates of change of g and m_1 are zero.

$$\dot{g} = -[\beta g^2 x^2 + \beta g m_1 x \pi_{m1} + \beta g m_2 x \pi_{m2} + g\gamma_g](1-\theta_g) + (m_1\gamma_{m1} + m_2\gamma_{m2})\theta_g \tag{3.17}$$

$$\dot{m}_1 = -m_1\gamma_{m1} + [\beta g^2 x^2 + g\gamma_g + \beta g m_1 x \pi_{m1} + \beta g m_2 x \pi_{m2} + m_1\gamma_{m1} + m_2\gamma_{m2}]\theta_{m1} \tag{3.18}$$

A dual medium of exchange equilibrium of this model is determined by the population fractions g and m_1 in equations (3.15) and (3.16) and by the production strategies θ_g and θ_{m1} computed from equations (3.17) and (3.18) in steady state. The equilibrium value functions in equations (3.12) to (3.14) can then be calculated.

Table 3.2 shows the results of some comparative statics for the equilibrium levels of goods inventory and money supplies. The direction of the changes in the population fractions was calculated assuming again that m_1 was the money with high production costs but low storage and depreciation costs. The level of inventories of goods did not respond to changes in any of the costs of producing or using goods. However, g was positively correlated with the costs for m_2 but negatively related to the costs for m_1. When money production costs became more similar (ie. c_{m2} rises or c_{m1} falls) then g rose but when storage or depreciation costs became more similar (ie. s_{m1} rises or s_{m2} falls) then g fell. The effects were reversed when costs became more dissimilar.

Table 3.2: Comparative Statics for Multiple Medium of Exchange

	g	m_1	m_2
c_g	0	+	-
c_{m1}	-	?	?
c_{m2}	+	?	?
s_g	0	+	-
s_{m1}	-	-	+
s_{m2}	+	?	-
γ_g	0	+	-
γ_{m1}	-	-	+
γ_{m2}	+	?	-
x	-	?	?

The high production cost money, m_1, was positively correlated with the costs associated with consumption goods. The response of m_1 to increases in the costs for the other type of money was ambiguous. When x was sufficiently close to 0 or 1 then m_1 fell when the other's costs increased. But when x was toward the middle of its range and utility was sufficiently larger than costs then the supply of m_1 increased in response to greater costs of the competing medium of exchange. Conversely, the supply of type one money fell as it became more costly to store or when it began to depreciate faster. The direction of change was ambiguous for changes in m_1's own production costs. However, regardless of x but for sufficiently large utility levels, the supply of type one money tended to decline when its own costs of production increased.

The pattern of response to parameter changes was basically the same for type two money. The only difference was that m_2 definitely did respond positively to increases in the storage or depreciation costs of the other type of money.

The model will quite readily generate equilibria with multiple types of media of exchange. There cannot however be a commodity that circulates as a medium of exchange and has strictly greater costs (production, storage, and depreciation) than another commodity which agents will also use as money. With two media of exchange, each money must have the lowest costs in at least one but not all categories. If any of the costs for a particular money rose high enough then that medium of exchange would be driven out of the economy in equilibrium. No steady state equilibrium could be maintained with positive amounts of that money being produced. Conversely, if the costs of one type of money fell far enough then it could drive the supply of the other medium of exchange to zero.

3.5 Fiat Money:

Now consider the model if there is an endogenous producible money called gold and a fiat currency offered in fixed supply by the government. As with goods and gold, agents may only hold one unit of the fiat currency at any given time. The fiat currency is assumed not to depreciate as does the producible money although it does possibly have a non-zero storage cost. Let V_f represent the value function for an agent with one unit of fiat currency in storage, f the supply of fiat money, and π_f and π_m the probabilities with which agents accept fiat and commodity money, respectively. The value functions for this form of the model can then be written as follows:

$$
\begin{aligned}
rV_g \;=\; & \beta g x^2 [\, u - \epsilon + \max_{\theta} \{\, \theta(V_g - c_g) + (1-\theta)(V_m - c_m) \,\} - V_g \,] \\
& + \beta m x \max_{\pi_m} \pi_m (V_m - V_g) \quad + \quad \beta f x \max_{\pi_f} \pi_f (V_f - V_g) \\
& + \gamma_g [\, \max_{\theta} \{\, \theta(V_g - c_g) + (1-\theta)(V_m - c_m) \,\} - V_g \,] - s_g
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
rV_m \;=\; & \beta g x \Pi_m [\, u - \epsilon + \max_{\theta} \{\, \theta(V_g - c_g) + (1-\theta)(V_m - c_m) \,\} - V_m \,] \\
& + \gamma_m [\, \max_{\theta} \{\, \theta(V_g - c_g) + (1-\theta)(V_m - c_m) \,\} - V_m \,] - s_m
\end{aligned}
\tag{3.20}
$$

$$
rV_f \;=\; \beta g x \Pi_f [\, u - \epsilon + \max_{\theta} \{\, \theta(V_g - c_g) + (1-\theta)(V_m - c_m) \,\} - V_f \,] - s_f
\tag{3.21}
$$

After consuming an agent can choose to produce either his production good with probability θ or a unit of the commodity money with probability 1-θ. Agents will be indifferent between the two production opportunities in equilibrium so that V_g-c_g = V_m-c_m. This still implies that a commodity money must have higher production costs than goods if it is to be valued in equilibrium. The value of accepting the fiat currency will be positive (V_f-V_g>0) in a monetary equilibrium but it is uncertain whether the value of having the fiat currency is greater or smaller than the value of a commodity money. A sufficient condition for $V_f > V_m$ is that the commodity money have higher storage costs which would be a reasonable assumption. Agents will not trade fiat money for commodity money. The only trading opportunity open to money holders is to find a suitable goods trader. Trading potential is basically identical for fiat and commodity money holders but commodity money traders face the risk of having their money depreciate while in storage.

Once again using the fact that agents are indifferent with respect to their production decisions, an expression defining the equilibrium fraction of goods traders can be derived from the value functions. Remembering that m=1-g-f, the steady state equilibrium condition for the number of goods traders is given by the following:

$$\begin{aligned}\{\, \beta gx(1-x)(u-\epsilon) + s_g - \beta mxc_m + (\beta gx^2 + \gamma_g + \beta mx)c_g \,\}\frac{r+\beta gx}{r+\beta x(g+f)} \\ + \{\, s_f + \beta gxc_g \,\}\frac{\beta fx}{r+\beta x(g+f)} - s_m + rc_g - [r + \beta gx + \gamma_m]c_m = 0\end{aligned} \tag{3.22}$$

The response of the number of goods traders to a change in the supply of fiat currency is uncertain. However, when fiat currency is first introduced to an economy operating at an equilibrium with a commodity money and population fractions defined by equation (3.6) then the following is true:

$$\left.\frac{dg}{df}\right|_{f=0} = \frac{\gamma_m c_m + s_m - s_f}{r(1-x)(u-\epsilon-c_g) + (r+\beta x)(c_m-c_g) + \gamma_m c_m - \gamma_g c_g + s_m - s_g} \tag{3.23}$$

A sufficient condition for this derivative to be positive is that commodity money storage and depreciation costs be greater than those costs for goods and fiat money. If this is true then the fraction of goods traders in the model will initially rise as a fiat currency is introduced. The fiat currency will initially only crowd out the commodity money. Because of the inventory restrictions on the storage of money and goods, however, further increases in the supply of fiat currency will eventually crowd out both goods and the commodity money. The introduction of fiat money tended to drive out commodity money much faster than goods. Agents have a certain level of need for media of exchange, whether in fiat or commodity form, so that more fiat money implies less commodity money.

Now consider the question of the effect of the fiat currency on the overall welfare of the agents. Aggregate welfare is defined again as the weighted average of the value functions for each state where the weights are the fractions of agents in each state. For a given level of the fiat currency the agents' optimal choice of goods inventories (g) is unchanged from the results for the model without fiat currency given in equation (3.10). The optimum is defined by the same level of goods inventories with and without fiat currency. The supply of commodity money is simply reduced by the amount of the fiat money. If the supply of government fiat currency becomes too high then optimal supply of the commodity money is driven to zero.

If the equilibrium level of g is determined from equation (3.22) and substituted into the welfare equation then we have welfare as a function of the supply of fiat money. Differentiating the resulting expression gives the following derivative:

$$\frac{drW}{df} = \frac{[\,(r(1-x)+\beta x)(u-\epsilon-c_g) - (r+\beta x)(c_m-c_g)\,][\,\gamma_m c_m + s_m - s_f\,]}{\Psi(f)} \qquad (3.24)$$

where $\Psi(f)$ is a complex polynomial in the supply of fiat money. Obviously there is no f which can set the derivative to zero and maximize welfare. Welfare will always be either increasing or decreasing in the supply of fiat money. However, evaluating the derivative at f=0 yields equation (3.25).

$$\frac{drW}{df}\bigg|_{f=0} = \frac{[\,(r(1-x)+\beta x)(u-\epsilon-c_g) - (r+\beta x)(c_m-c_g)\,][\,\gamma_m c_m + s_m - s_f\,]}{r(1-x)(u-\epsilon-c_g) + (r+\beta x)(c_m-c_g) + \gamma_m c_m - \gamma_g c_g + s_w - s_g} \quad (3.25)$$

The change in welfare when money is first introduced can be shown to be positive provided $\gamma_m c_m + s_m - s_f > 0$ which is likely true since one would expect it to cost more to store, the commodity money than the fiat money. Using equation (3.22), we see that the expression $[r(1-x)+\beta x](u-\varepsilon-c_g)-(r+\beta x)(c_m-c_g)$ and the denominator in equation (3.25) must be positive or else the equilibrium goods inventories will be outside the unit interval. When welfare is increasing in the supply of fiat money then the government will want to raise that supply at least until that point at which the supply of the commodity money is driven to zero. This is likely to happen before the fiat money supply reaches its maximum. The responsiveness of welfare to fiat money will change at that point once the economy moves into a non-commodity money equilibrium.

The welfare maximizing level of fiat money in an equilibrium without commodity money will likely be less than that level which drives the commodity money out of the economy. In the presence of a commodity money, the optimal government strategy may be to raise the supply of fiat currency until the private money is driven out and then remove some of the fiat money until welfare is maximized. However, for large values of x implying less of a need for money because trade is easier, it will be optimal to completely remo· the fiat money as well once agents have set $\pi_m=0$ and will no longer accept the commodity money as a medium of exchange. For smaller numbers of trading partners it will be optimal to leave a certain quantity of fiat money in circulation.

Suppose the government outlaws the use of private commodity money (ie.

increases c_m so it is prohibitively high to produce) and then introduces a fiat money to replace it. If the supply of fiat money injected is equal to the commodity money previously circulating in equilibrium then the agents' welfare will increase provided $\gamma_m c_m + s_m - s_f > 0$ which is expected to be true. Obviously, setting the fiat money supply at its welfare maximizing level will improve the agents' situation even further. If, however, no fiat money or only a small amount of it replaces the commodity money then it is possible that aggregate welfare will be reduced. This occurred if the commodity money equilibrium provided higher welfare than the pure barter equilibrium. Therefore, agents benefit from the removal of the commodity money only if it is replaced with sufficiently high quantities of the fiat currency. If x was high and the commodity money actually decreased welfare then the government need not replace the commodity money with a fiat money in order to increase welfare.

The introduction of a fiat money will not stop agents from producing and using their old commodity money provided of course that not too much fiat currency is introduced. The commodity money can be driven out of the economy if sufficiently large levels of fiat currency are injected into circulation. The overall welfare of the agents will be increased by the introduction of the fiat money because it replaces a commodity money which is costly to produce. If the commodity money is removed from circulation by a law then welfare will be increased only if sufficient quantities of fiat money are injected to serve as a new medium of exchange.

3.6 Conclusion:

Given the opportunity to produce their own money the agents in this economy chose to overproduce money (for a wide set of parameter values) to such an extent that it could substantially lower their overall welfare. The producers fail to account for the externality that their behaviour while producing affects their future returns once they became goods or money traders. The production of money was too high and goods too low for the agents to be able to maximize their overall welfare. The supply of money produced by the agents increased as the properties of the commodity money improved, or in other words, as the costs of its use declined.

A commodity must be sufficiently scarce or sufficiently difficult to produce if it is to be valued and serve as a medium of exchange in this environment. Conversely, the production, depreciation, and storage costs of money also cannot be too high or else no agent would willingly use it as a medium of exchange. It is possible however to have a commodity money circulating as a generally accepted medium of exchange in equilibrium even if it has strictly greater costs than the consumption goods. The benefits of faster trade can outweigh the higher user costs. The optimal level of commodity money was positive under certain conditions, a significant double coincidence of wants problem. An equilibrium with commodity money could also yield greater aggregate welfare than a pure barter equilibrium.

The model was able to generate equilibria with multiple types of money being used by the agents. However, there could be no endogenous medium of exchange which had strictly greater costs of each type than another medium of exchange. Agents would choose not to use such a commodity as a money. If the characteristics of one type of money were shocked so that its depreciation rate or storage cost decreased by a sufficient amount, then all other types of private money could be driven out of the economy. Similarly, if the production costs for one type of money became sufficiently small relative to the production costs for goods or other monies then the agents might cease to use the other money in their trading.

The introduction of an exogenous government fiat money into the economy could only drive out the commodity money if there was a sufficiently large supply of the fiat currency. This was true even if the fiat currency had strictly lower costs than any commodity money. The introduction of some fiat currency would, however, increase the overall welfare of the agents. The government could increase the agents' welfare by outlawing the production of commodity money and replacing it with sufficiently large quantities of fiat money. The agents still had use of a generally accepted medium of exchange but were no longer required to pay a production cost to acquire it.

The money supply and goods output may be positively or negatively correlated in response to changes in the parameters of the model. The correlation was positive only if the initial level of goods inventories was sufficiently high.

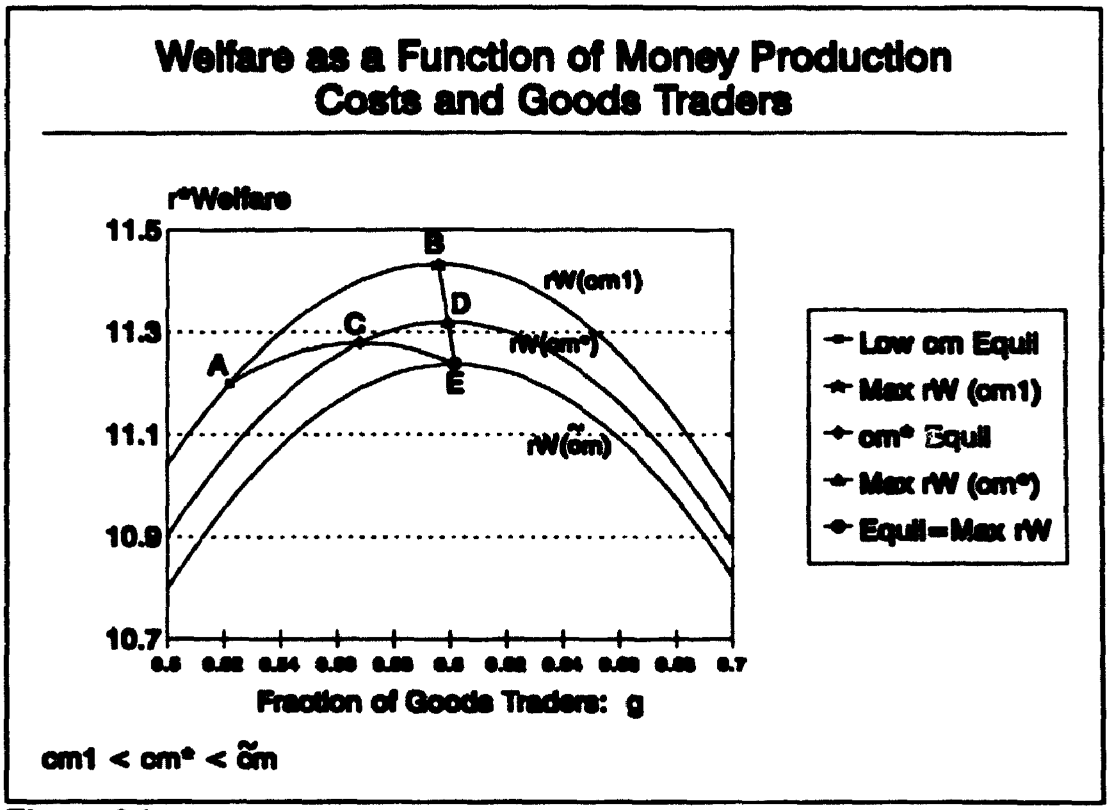
Welfare as a Function of Money Production
Costs and Goods Traders
r*Welfare
11.5
11.3
11.1
10.9
10.7
B
rW(om1)
D
C
rW(om*)
A
E
rW(õm)
Low om Equil
Max rW (om1)
om* Equil
Max rW (om*)
Equil=Max rW
Fraction of Goods Traders: g
om1 < om* < õm

Figure 3.1

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